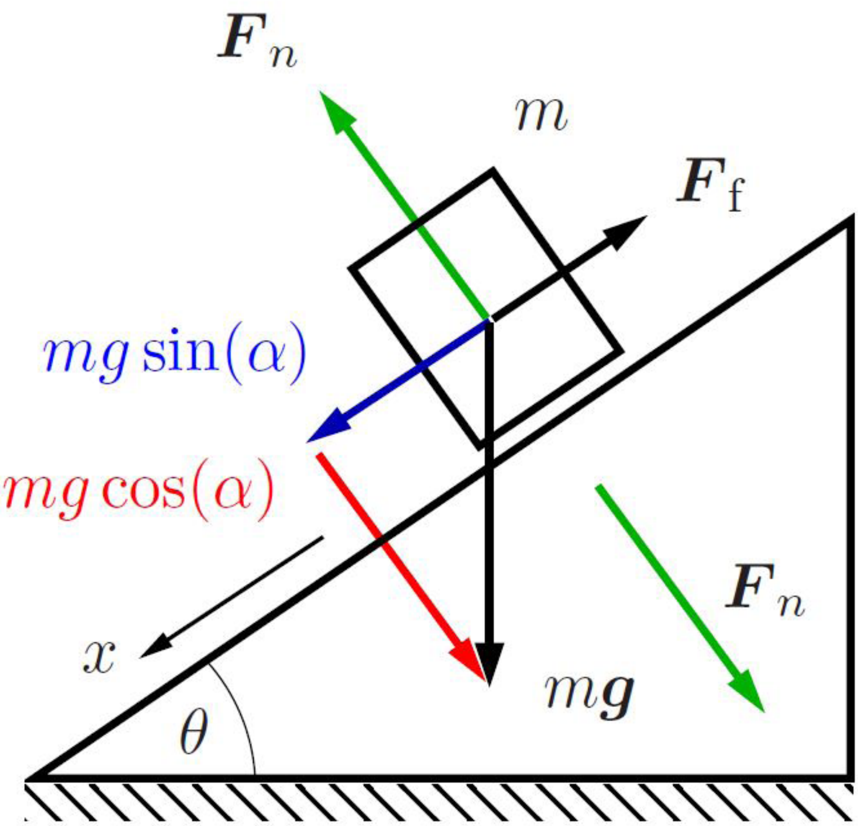
Basic Laws, Equations and Models I.



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Gergely Nyitray

Basic Laws, Equations and Models I.

Pécs

2019

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MODEL FOR VALUE-ORIENTED, OPENNESS AND INCLUSIVE
APPROACHES IN 21. CENTURY HIGHER EDUCATION

Fundamental Laws, Equations and Models I

GENERAL COURSE
MECHANICS

BY
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PÉCS
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Introduction

Physics is the branch of science that describes matter, energy, space, and time at the most fundamental level. Whether you are planning to study biology, architecture, medicine, music, chemistry, or art, there are principles of physics that are relevant to your field. The goal is to find the most basic laws that govern the universe and to formulate those laws in the most precise way possible.

The study of physics is valuable for several reasons:

- Since physics describes matter and its basic interactions, all natural sciences are built on the foundations of the laws of physics.
- In today's technological world, many important devices can be understood correctly only with a knowledge of the underlying physics.
- By studying physics, you acquire skills that are useful in other disciplines.
- Physics is an exciting intellectual adventure that inspires young people and expands the frontiers of our knowledge about Nature.

Classical mechanics deals with the question of how an object moves when it is subjected to various forces, and also with the question of what forces act on an object which is not moving. The word "classical" indicates that we are not discussing phenomena on the atomic scale and we are not discussing situations in which an object moves with a velocity which is an appreciable fraction of the velocity of light. The laws of classical mechanics enable us to calculate the trajectories of bullets, space vehicles, and planets as they move around the sun. Using these laws we can predict the position-versus-time relation for a cylinder rolling down an inclined plane or for an oscillating pendulum. Anyone who seriously studies mechanics, even at an elementary level, will find the experience a true intellectual adventure.

The course is a result of fourteen year's work in the faculty of engineering of the University of Pécs. While using this book, try not to memorize the material formalistically and mechanically, but logically. Please memorize the material by thoroughly understanding it. I have tried to present physics not as a certain volume of information to be memorized, but as clever, logical, and attractive science. It is left to the reader to judge the extent to which I have succeeded in doing this.

I have done my utmost to limit the size of the course. This was achieved by carefully choosing the material which in my opinion should be included in a general course of physics. It's important to note that I have used the experiences of several books, lectures and online courses. They are listed at the end of this note (Bibliography).

The author is grateful to Prof. Dr. László Pálfalvi and to Tímea Györök for careful reading of the manuscript and valuable suggestions and comments.

Chapter 1

Physics in General

1.1 General Remarks on Physics

Physics is a science dealing with the most general properties and forms of motion of matter. It concentrates knowledges accumulated on the most general properties and phenomena of the world surrounding us. The fundamental method of investigation in physics is the running of an *experiment*, i. e. the observation of the phenomenon being studied in accurately controlled conditions. The *laws* of physics are established by generalizing experimental facts. They express the objective regularities existing in nature. These laws are expressed in the form of quantitative relationships between various physical quantities. It is very important to understand that the laws of physics (unlike mathematics) are not absolute laws. Any physical law has got qualified clauses, which means it is valid only in some regions or regimes of physical parameters. Only for certain ranges of variation of physical quantities are these laws valid. For this reason most of the laws of physics have to be constantly refined.

Efforts are being made to unify these laws and reduce the number of independent laws. The most sophisticated form of unification of laws is called *variation principles*. The first variation principle in the history of physics was the Fermat's principle. According to this principle the path taken between two points by a ray of light is the path that can be traversed in the least time. Fermat's principle can be used to describe the properties of light rays reflected off mirrors, refracted through different media, or undergoing total internal reflection. This principle has unified the laws of geometric optics. The most important variations principles of mechanics are the *principle of least action* (Hamilton's principle) and the Gauss's *principle of least constraint*.

Hypotheses are enlisted to explain experimental data. A hypothesis is a scientific assumption advanced to explain a definite fact or phenomenon and requiring verification and proving to become a scientific a scientific theory or law. A *physical theory* is a system of basic ideas summarizing experimental data and reflecting the objective regularities of nature. The most sophisticated form of theories is the *unified theory*. Physics is subdivieded into so-called classical physics and quantum physics.

1.2 The Edifice of Physics

The edifice of classical physics built up by the end of the 19th century was very harmonious. Most physicists were convinced that they already knew everything about nature that could be known. The formal scientific definition of "theory" is quite different from the everyday meaning of the word. It refers to a comprehensive explanation of some aspects of nature that is supported by a vast body of evidence. Many scientific theories are so well established that no new evidence is likely to alter them substantially. For example, no new evidence will demonstrate that the Earth does not orbit around the sun (heliocentric theory), or that living things are not made up of cells (cell theory), that matter is not composed of atoms, or that the surface of the Earth is not divided into solid plates that have moved over geological timescales (the theory of plate tectonics)... One of the most useful properties of scientific theories is that they can be used to make predictions about natural events or phenomena that have not yet been observed.

In physics the term theory is generally used for a mathematical framework—derived from a small set of basic postulates (usually symmetries, like equality of locations in space or in time, or identity of electrons, etc.)—which is capable of producing experimental predictions for a given category of physical systems. One good example is classical electromagnetism which is based on a few equations called Maxwell's equations. The specific mathematical aspects of classical electromagnetic theory are termed "laws of electromagnetism", reflecting the level of consistent and reproducible evidence that supports them. Within electromagnetic theory generally, there are numerous hypotheses about how electromagnetism applies to specific situations. Many of these hypotheses are already considered adequately tested, with new ones always in the making and perhaps untested.

1.2.1 Scientific Advancement

A scientific theory is a well-substantiated explanation of some aspects of the natural world, based on a body of facts that have been repeatedly confirmed through observation and experiment. Such fact-supported theories are not "guesses" but reliable accounts of the real world. The theory of biological evolution is more than "just a theory". It is as factual an explanation of the universe as the atomic theory of matter or the germ theory of disease. Our understanding of gravity is still a work in progress. But the phenomenon of gravity, like evolution, is an accepted fact. A theory of everything (ToE) is a hypothetical single, all-encompassing, coherent theoretical framework of physics that fully explains and links together all physical aspects of the universe as it can be seen in figure (1.1). Finding a ToE is one of the major unsolved problems in physics. Over the past few centuries, two theoretical frameworks have been developed that, as a whole, most closely resemble a ToE. These two theories upon which all modern physics rests are General Relativity (GR) and quantum field theory (QFT). GR is a theoretical framework that only focuses on gravity for understanding the universe in regions of both large scale and high mass: stars, galaxies, clusters of galaxies, etc. On the other hand, QFT is a theoretical framework that only focuses on

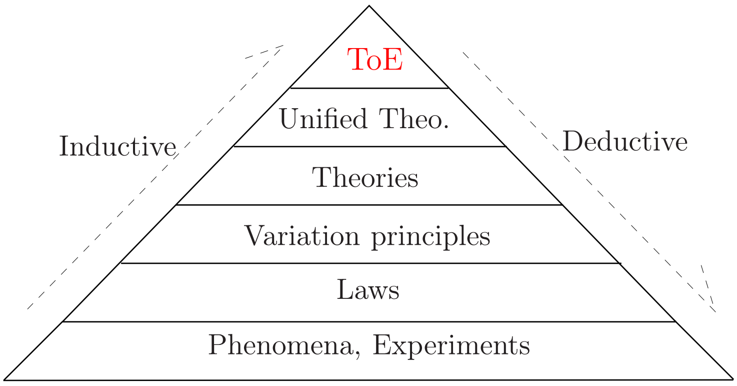


Figure 1.1: Hierarchy of the most important notions in Physics.

three non-gravitational forces for understanding the universe in regions of both small scale and low mass: sub-atomic particles, atoms, molecules, etc. QFT successfully implemented the Standard Model and unified the interactions (so-called Grand Unified Theory) between the three non-gravitational forces: weak, strong, and electromagnetic force.

1.3 Mechanics

Mechanics is the foundation of pure and applied sciences. Its principles apply to the vast range and variety of physical systems. The aim of classical mechanics is to understand physical phenomena and laws of mechanics and to apply them to different, everyday situations. In order to achieve this it is necessary to perform mathematical calculations and solve problems.

Mechanics is one of the oldest and most familiar branches of physics. It deals with bodies at rest and in motion and the conditions of rest and motions when bodies are under the influence of internal and external forces. The study of mechanics may be divided into two parts: kinematics and dynamics. Kinematics is concerned with a purely geometrical description of the motion, disregarding the forces producing the motion. It deals with concepts and the interrelations between position, velocity, acceleration and time. Dynamics is concerned with the forces that produce changes in motion or changes in other properties, such as the shape and size of objects. Statics deals with bodies at rest under the influence of external forces.

1.4 Units and Dimensions

The laws of physics establish quantitative relations between physical quantities. To establish such relations, it is necessary to be able to measure various physical quantities. To measure a physical quantity means to compare it with a quantity of the same kind taken as a unit. For this reason in physics, a number to specify a quantity is useless unless we know the unit attached to the number. Generally speaking, we could establish a unit for every physical quantity arbitrarily. We can limit ourselves, however, to an arbitrary choice of the units for only three quantities taken as the basic ones.

There are several systems differing in the selection of the basic units. Systems based on the units of length, mass, and time are called *absolute*.

$$\begin{aligned}\text{length } [\mathrm{l}] &= [\mathrm{m}]\\ \text{mass } [\mathrm{m}] &= [\mathrm{kg}]\\ \text{time } [\mathrm{t}] &= [\mathrm{s}]\\ \text{electric current } [\mathrm{I}] &= [\mathrm{A}]\\ \text{thermodynamic temperature } [\mathrm{T}] &= [\mathrm{K}]\\ \text{amount of substance } [\mathrm{n}] &= [\mathrm{mol}]\\ \text{luminous intensity } [\mathrm{I}_v] &= [\mathrm{cd}]\end{aligned}$$

Derived units are constructed from combinations of the base units.

The relation showing how a unit of quantity changes when the basic units are changed is called the *dimension* of this quantity. Many different units of length exist: meters, inches, miles, nautical miles, astronomical units, just to name a few. All have the dimensions of length; each can be converted into any other. *We can add, subtract, or equate quantities only if they have the same dimensions.* The dimension is designated by its symbol placed in brackets. Special symbols are used for the dimensions of the basic quantities: L for length, M for mass, and T for time. When these symbols are used, the dimension of an arbitrary physical quantity has the form $\mathrm{L}^\alpha\mathrm{M}^\beta\mathrm{T}^\gamma$. α, β, and γ may be either positive or negative, and in particular may equal zero.

1.4.1 Dimensional Analysis

Since physical laws cannot depend on the selection of the units for the quantities figuring in them, the dimensions of both sides of the equations expressing these laws must be the same. This condition can be used first, for verifying the correctness the physical relations obtained, and second, for establishing the dimensions of physical quantities.

$$\begin{aligned}[v] &= \mathrm{LT}^{-1}\\ [a] &= \mathrm{LT}^{-2}\\ [F] &= \mathrm{MLT}^{-2}\end{aligned}$$

In some cases, we can completely solve a problem—up to a dimensionless factor—using dimensional analysis. To do this, first list all relevant quantities upon which the answer might depend. Then determine what combinations of them have the same dimensions as the answer for which we are looking. If only one such combination exists, then we have the answer, except for a possible dimensionless multiplicative constant.

Let's look at an example. An ideal LC-circuit (see figure (1.2)) is an oscillatory system. What is the time period of the circuit? We assume that the time period of

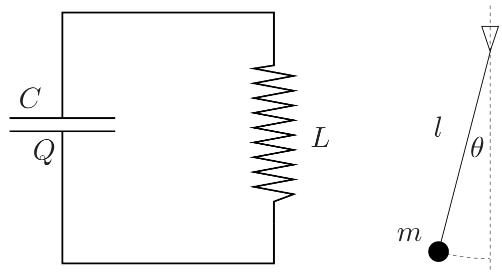


Figure 1.2: An LC circuit (on the left) and a pendulum (on the right)

the circuit is characterized by the inductance L and the capacity C. Hence,

$$L^{\alpha}C^{\beta} = T^1$$

If we know the definitions of L and C the units and dimensions can be easily determined

$$[C] = \frac{[Q]}{[U]} = \frac{[I][t]}{[W][Q]^{-1}} = \frac{\mathrm{A}^2\mathrm{s}^4}{\mathrm{kg\,m}^2}$$
$$[L] = \frac{[\phi]}{[I]} = \frac{[B][A]}{[I]} = \frac{\mathrm{kg\,m}^2}{\mathrm{s}^2\mathrm{A}^2}$$

We used $[W] = \mathrm{kg\,m^2s^{-2}}$ and $[B][A] = \mathrm{kg\,m^2s^{-2}A^{-1}}$. It is worth mentioning that $[A]$ means the unit of an area (m^2) and A^{-1} stands for the unit of electric current (ampere) to the power of -1. Both sides of the equation have dimensions of time, the equation is dimensionally consistent.

$$\begin{aligned}
T^1 &= L^{\alpha}C^{\beta} \\
\mathrm{T}^1 &= [\mathrm{I}^2\,\mathrm{T}^4\,\mathrm{M}^{-1}\,\mathrm{L}^{-2}]^{\alpha}\cdot[\mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-2}\,\mathrm{I}^{-2}]^{\beta} \\
\mathrm{T}^1 &= [\mathrm{I}^{2\alpha}\,\mathrm{T}^{4\alpha}\,\mathrm{M}^{-\alpha}\,\mathrm{L}^{-2\alpha}\,\mathrm{M}^{\beta}\,\mathrm{L}^{2\beta}\,\mathrm{T}^{-2\beta}\,\mathrm{I}^{-2\beta} \\
\mathrm{T}^1 &= \mathrm{I}^{2\alpha-2\beta}\cdot\mathrm{T}^{4\alpha-2\beta}\cdot\mathrm{M}^{\beta-\alpha}\cdot\mathrm{L}^{2\beta-2\alpha}
\end{aligned}$$

I, M, an L must be raised to the power of zero, while the power of T must be 1, hence

$$\begin{aligned}
2\alpha - 2\beta = 0 \rightarrow &\quad \alpha = \beta \\
4\alpha - 2\beta = 1 \rightarrow &\quad \alpha = \frac{1}{2}
\end{aligned}$$

Hence the time period of the oscillations is

$$T = L^{1/2}C^{1/2} = \sqrt{LC}$$

The exact value of $T = 2\pi\sqrt{LC}$, where 2π is a dimensionless factor.

A physical pendulum is simply a rigid object which swings freely about some pivot point. The physical pendulum may be compared with a simple pendulum, which consists of a small mass suspended by a (ideally massless) string. What is the time period of a simple pendulum? We assume that the time period depends on g and l. The units and dimensions of such quantities are the following:

$$[g] = \frac{\mathrm{m}}{\mathrm{s}^2} \quad [l] = \mathrm{m} \text{ units}$$
$$[g] = \frac{\mathrm{L}}{\mathrm{T}^2} = \mathrm{LT}^{-2} \quad [l] = \mathrm{L} \text{ dimensions}$$

$$\begin{aligned}
T^1 &= g^{\alpha}\, l^{\beta} \\
\mathrm{T}^1 &= \left(\mathrm{LT}^{-2}\right)^{\alpha} \mathrm{L}^{\beta} \\
\mathrm{T}^1 &= \mathrm{L}^{\alpha+\beta}\mathrm{T}^{-2\alpha} \\
-2\alpha &= 1 \rightarrow \alpha = -\frac{1}{2} \\
0 &= \alpha + \beta \rightarrow \beta = \frac{1}{2}
\end{aligned}$$

Finally the time period yields:

$$\begin{aligned}
T &= g^{-\frac{1}{2}}\, l^{\frac{1}{2}} \\
T &= \sqrt{\frac{l}{g}}
\end{aligned}$$

The exact value of $T = 2\pi\sqrt{l/g}$, where 2π is a dimensionless factor.

1.4.2 Planck-time, Planck-length and Planck-mass

The term ‘fundamental physical’ constant is often used to refer to the dimensionless constants, but has also been used to refer to certain universal dimensioned physical constants, such as the speed of light c, vacuum permittivity ε_0, Planck constant h, and the gravitational constant G, that appear in the most basic theories of physics. In 1899 Max Planck suggested that there existed some fundamental natural units for length, mass, time and energy. These he derived applying dimensional analysis, using only the Newton gravitational constant, the speed of light and the Planck constant. The natural units he derived later became known as “the Planck length”, “the Planck mass”, “the Planck time” and “the Planck energy”. They take the following forms:

$$\begin{aligned}
t_{\mathrm{P}} &= \sqrt{\frac{\hbar G}{c^5}} \approx 5.391 \cdot 10^{-44}\,\mathrm{s} \\
l_{\mathrm{P}} &= \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \cdot 10^{-35}\,\mathrm{m} \\
m_{\mathrm{P}} &= \sqrt{\frac{\hbar c}{G}} \approx 2.176 \cdot 10^{-8}\,\mathrm{kg}
\end{aligned}$$

where $\hbar = h/2\pi$ by definition. Fundamental physics is concerned with the ultimate microscopic laws of nature. In our current understanding, these laws describe gravity according to Einstein's general theory of relativity, and everything else according to the Standard Model of particle physics. With just a few exceptions (dark matter and dark energy, neutrino masses), these two theories are consistent with all the experimental data we have. The bad news is that they are mutually inconsistent. The Standard Model is a quantum field theory, a direct outgrowth of the quantum-mechanical revolution of the 1920's. General Relativity, meanwhile, remains a classical theory, very much in the tradition of Newtonian mechanics. The program of "quantum gravity" is to invent a quantum-mechanical theory that reduces to General Relativity in the classical limit. This is obviously a crucially important problem, but one that has traditionally been a sidelight in the world of theoretical physics. For one thing, coming up with good models of quantum gravity has turned out to be extremely difficult; for another, the weakness of gravity implies that quantum effects don't become important in any realistic experiment. There is a severe conceptual divide between General Relativity and the Standard Model, but as a practical matter there is no pressing empirical question that one or the other of them cannot answer.

Chapter 2

Kinematics

Mechanical motion is the simplest form of motion of matter. We can see movements of bodies everywhere in our ordinary life. This is why mechanical notions are so clear. This also explains the fact that mechanics was the first of all the natural sciences to be developed very broadly.

A combination of bodies separated for consideration is called a *mechanical system*. The bodies to be included in a system depend on the nature of the problem being solved. In a particular case, the system may consist of a single body. Motion occurs both in space and in time. Consequently, to describe motion, we must also determine time. We use watch or clock for this purpose. A combination of bodies that are stationary relative to one another with respect to which motion is being considered forms a *reference frame*. To describe the motion of a body means to indicate for every moment of time the position of the body in space and its velocity.

The plane of Kinematics is to predict the future given by the present. What you need to know is the initial location of the object and the initial velocity.

2.1 Motion in 1D

What we are going to study is a non-living object and we are going to pick it to be a mathematical point. A body whose dimensions may be disregarded in the conditions of a given problem is called a *point particle*. You can imagine a bead with wire going through it and the bead can only slide back and forth (see figure (2.1)). We pick an

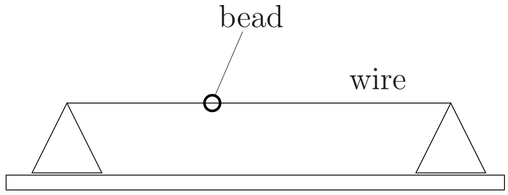


Figure 2.1: One-dimensional "world".

origin, call it zero, we put some markers there to measure distance, and we say this bead is sitting here at $x = x_0$. As time goes by the position of the bead is unchanged

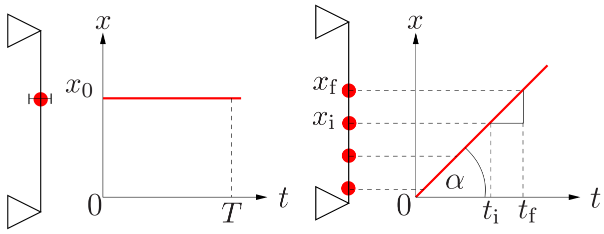


Figure 2.2: On the left: the position-time graph of a stationary object. On the right: the position-time graph of a uniform motion along a straight line.

as it can be seen in figure (2.2). Position-time graph for an object in *uniform motion* along a straight path is a straight line inclined to the time axis. The velocity of a uniform motion is equal to the slope of position-time graph

$$\langle v \rangle = \tan(\alpha) = \frac{\Delta x}{\Delta t} \tag{2.1}$$

In case of motions with changing velocity this expression can be considered as the definition of the average velocity. In case of uniform motion along a straight line the instantaneous velocity and the average velocity are the same. It is worth mentioning that the symbol Δ does not stand alone and cannot be cancelled in equations because it modifies the quantity. The symbol Δ is a capital Greek letter that stands for "subtract". For quantity x if we need to find the change of position of a particle, we subtract them $\Delta x = x_{\text{final}} - x_{\text{initial}}$.

The average velocity can be written in the following form:

$$\langle v \rangle = \frac{x_{\text{f}} - x_{\text{i}}}{t_{\text{f}} - t_{\text{i}}} = \frac{x(t + \Delta t) - x(t)}{\Delta t} \tag{2.2}$$

2.1.1 Instantaneous Velocity

The velocity $\boldsymbol{v}$ of a particle can change with time both in magnitude and in direction. The rate of change of the magnitude of the velocity vector is determined by the derivative of $v(t)$ with respect to t. The instantaneous velocity can be used to calculate the displacement of the object during a very short time interval.

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \langle v \rangle = \left. \frac{d\,x(t')}{dt'} \right|_{t'=t} \tag{2.3}$$

To find the instantaneous velocity at some time $t = t'$, we draw lines showing the average velocity for smaller and smaller intervals. As the time intervals are reduced, the average velocity changes. As it can be seen in the figure (2.3) as Δt gets smaller and smaller, the chord approaches a tangent line to the graph at t'. *Thus, v is the slope of the line tangent to the graph of $x(t)$ at a chosen time.*

The notation $\lim\limits_{\Delta t \to 0}$ is read "the limit, as Δt approaches zero, of..." Rate of change of any q quantity in physics is basically the differentiation of that quantity with respect to time.

$$\frac{dq}{dt} = \dot{q}(t)$$

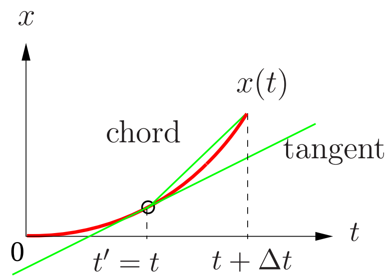


Figure 2.3: Geometric meaning of the instantaneous velocity: the slope of the line tangent.

The traditional notation has been used of replacing d/dt, the derivative taken along the actual particle trajectory, by an overdot. In case of motion in higher dimension (2D and 3D) the vector feature of velocity becomes more and more important. The instantaneous velocity $\boldsymbol{v}$ *is a vector quantity whose magnitude is the speed and whose direction is the direction of motion.*

What is the instantaneous velocity during free fall? If air resistance is negligible, the only appreciable force is that of gravity. In free fall, no forces act on an object other than the gravitational force that makes the object fall. In the 16th century Galileo realized that during free fall the displacement of a body is proportional to the time squared. Let us calculate the instantaneous velocity of the falling object if we know the definition of derivative respect to time

$$\frac{d\,x(t)}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t},$$

and the Galileo's rule $x(t) = (g/2)\,t^2$ for falling objects. The initial velocity is assumed to be zero hence the tangent line of the parabola must be parallel to the horizontal axis at $t = 0$. *It is important to emphasize that the $x - t$ graph (figure 2.4) shows a*

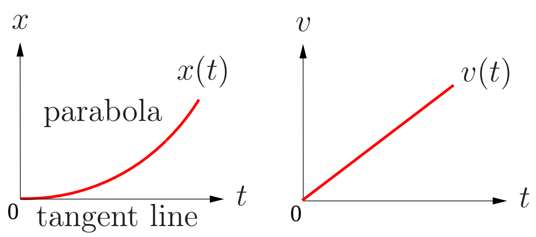


Figure 2.4: On the left: position-time graph of a body during free fall. On the right: velocity-time graph of the same particle.

curving line, but that does not mean the particle travels along a curved path!

According to the concept of the instantaneous velocity

$$
\begin{aligned}
v &= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \\
v &= \lim_{\Delta t \to 0} \frac{\frac{g}{2}(t+\Delta t)^2 - \frac{g}{2}(t)^2}{\Delta t} \\
v &= \frac{g}{2} \lim_{\Delta t \to 0} \frac{\cancel{t^2} + 2t\cancel{\Delta t} + \Delta t^{\cancel{2}} - \cancel{t^2}}{\cancel{\Delta t}} \\
v &= \frac{g}{2} \lim_{\Delta t \to 0} (2t + \Delta t) = g\,t
\end{aligned}
$$

In line with experience we get that the instantaneous velocity is proportional to t.

2.1.2 Instantaneous Acceleration

What is the rate of change of the velocity during free fall? The rate of change of the velocity is called acceleration (symbol $\boldsymbol{a}$). The definition of the average and instantaneous acceleration is very similar to the definition of average and instantaneous velocity.

$$
\begin{aligned}
\langle a \rangle &= \frac{v(t+\Delta t) - v(t)}{\Delta t} \\
a &= \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}
\end{aligned}
$$

We showed that $v(t) = g\,t$, hence

$$
a = \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - gt}{\Delta t} = \frac{\cancel{gt} + g\cancel{\Delta t} - \cancel{gt}}{\cancel{\Delta t}} = g
$$

In line with experience we get that the acceleration is constant $g \approx 9.81$ m/s^2.

2.1.3 Kinematic Equations in 1D

The following kinematic formulas are very useful to solve problems in 1D, however they only work for time intervals of constant acceleration.

$$
\begin{aligned}
v(t) &= v_0 + a\,t && (2.4) \\
x(t) &= x_0 + v_0 t + \frac{a}{2} t^2 && (2.5) \\
\Delta x &= \left(\frac{v + v_0}{2} \right) t && (2.6) \\
v^2 &= {v_0}^2 + 2a(x - x_0) && (2.7)
\end{aligned}
$$

Since the kinematic equations are only accurate if the acceleration is constant during the time interval considered, we have to be careful not to use them when the acceleration is changing (simple harmonic motion). Also, the kinematic formulas assume all variables are referring to the same direction.

A kinematics problem begins by describing the geometry of the system and declaring the initial conditions of any known values of position, velocity and/or acceleration of points within the system. The x_0 and v_0 are the initial conditions of the system. The goal of Kinematics is to predict the future given by the initial conditions. In the framework of classical kinematics the future is predictable in any degree of accuracy.

2.1.4 A Car and a Constant Breaking Force

A car of mass m moves along a horizontal road with uniform motion and speed v_0. At time $t = 0$ s a constant braking force starts acting on it. Assume that at time $t = 0$ s the coordinate x equals zero. Let us determine how the velocity $v(t)$ and the coordinate $x(t)$ are changing with time and the time T at which the car stops, and the distance S which the car travels during stopping.

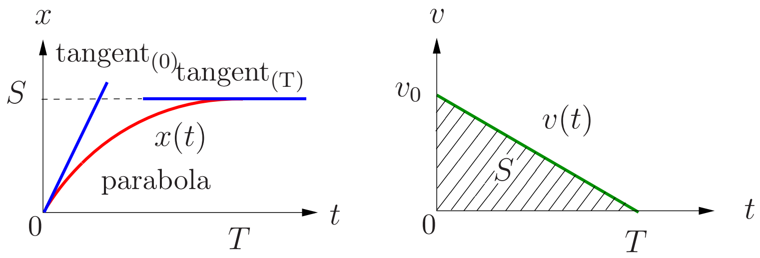


Figure 2.5: Position-time and velocity-time graphs of a slowing object with constant acceleration.

In accordance with figure (2.5) the line tangents at $t = 0$ s and at $t = T$ are the following $x(t)_0^{\tan} = v_0\,t$ and $x(t)_T^{\tan} = S$. According to the first kinematic equation

$$\begin{aligned} v(t) &= v_0 - a\,t \\ 0 &= v_0 - a\,t \end{aligned}$$

we get the breaking time T

$$T = \frac{v_0}{a}$$

We declaire that the displacement Δx during any time interval equals the area under the graph of $v(t)$ (see figure 2.5). When we speak of the area under the graph, we are not talking about the literal number of square centimetres. According to this new information the breaking path can be easily calculated:

$$S = \frac{v_0 T}{2} = \frac{v_0^2}{2a}$$

Identical result can be obtained from the equation of

$$S = v_0\,t - \frac{a}{2}t^2 = v_0\,T - \frac{a}{2}\,T^2 = v_0\frac{v_0}{a} - \frac{\cancel{a}}{2}\frac{v_0^2}{a^{\cancel{2}}} = \frac{v_0^2}{2a}$$

2.1.5 Projectile Motion Vertically Upwards

Let us consider an object moves with constant acceleration in the Earth's gravitational field vertically upward. The velocity decreases linearly from its initial value. Then v continues to decrease at the same rate and is now negative with its magnitude getting larger and larger. At $t = T$, when the projectile has returned to its original altitude, the velocity has the same magnitude as at $t = 0$ but the opposite sign ($v = -v_0$).

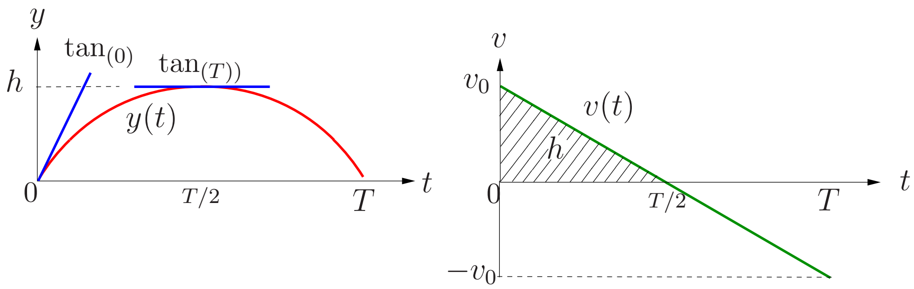


Figure 2.6: Position-time and velocity-time graphs of a projectile. The projectile was launched upward.

The graph of $y(t)$ indicates that the projectile moves upward, quickly at first and gradually slowing, until it reaches the maximum height (see figure (2.6)).

$$y(t) = v_0\, t - \frac{g}{2}\, t^2$$
$$v(t) = v_0 - g\, t$$

2.2 Motions in 2D

2.2.1 Motion of Projectiles

If an object moves in the xy-plane with constant acceleration, then both a_x and a_y are constant. By looking separately at the motion along two perpendicular axes, the y-direction and the x-direction, each component becomes a one-dimensional problem, which we already know how to solve. We can apply any of the constant acceleration relationships separately to the x-components and to the y-components.

$$\begin{array}{lll} a_x = 0 & v_x = v_{0x} & x(t) = v_{0x}\, t \\ a_y = -g & v_y = v_{0x} - gt & y(t) = v_{0y}\, t - \frac{g}{2} t^2 \end{array}$$

We choose the axes so that the acceleration is in the positive or negative y-direction. Then $a_x = 0$ and v_x is constant.

An object in free fall near the Earth's surface has a constant acceleration. As long as air resistance is negligible, the constant downward pull of gravity gives the object a constant downward acceleration equal to $\boldsymbol{g}$. As it can be seen in figure (2.7) now we consider objects (called projectiles) in free fall that have a *nonzero* horizontal velocity component.

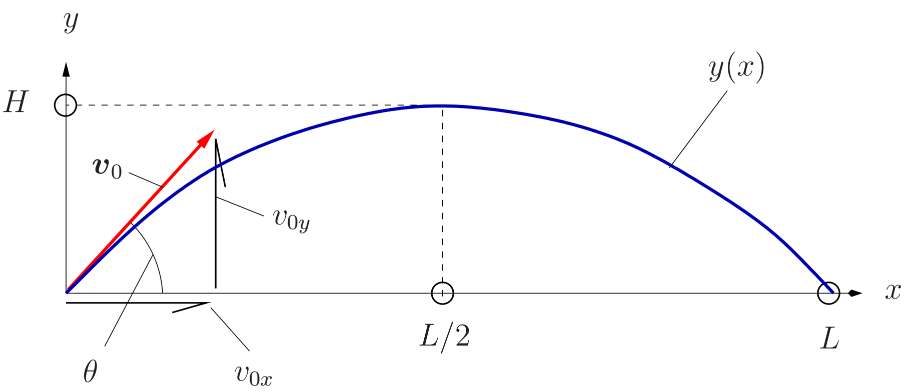


Figure 2.7: Position-position graphs of a projectile.

The initial velocity $\boldsymbol{v}_0$ is at an angle θ above the horizontal, then resolving it into components gives

$$v_{0x} = v_0 \cos(\theta) \tag{2.8}$$
$$v_{0y} = v_0 \sin(\theta) \tag{2.9}$$

Since the gravitational force pulls in the $-y$-direction, the horizontal component of the net force is zero. Therefore, $a_x = 0$ and the horizontal velocity component v_x is *constant*. The vertical velocity components v_y change at a constant rate.

At the peak point of the trajectory the v_y must be zero, hence

$$0 = v_{0y} - gt \rightarrow \tau = \frac{v_{0y}}{g}$$

τ is often referred to as half time of flight $(T/2)$. By substituting for t with τ in position-time function $y(\tau) = v_{0y}\,\tau - \frac{g}{2}\tau^2$ we obtain the height H of the projectile as follows:

$$H = v_{0y}\frac{v_{0y}}{g} - \frac{\cancel{g}}{2}\frac{v_{0x}^2}{g^{\cancel{2}}} = \frac{v_{0y}^2}{2g} = \frac{v_0^2 \sin(\theta)}{2g}$$

The range L of the motion can be calculated as follows:

$$L = v_{0x} \cdot (2\tau) = v_0 \cos(\theta)\, 2\frac{v_0 \sin(\theta)}{g} = \frac{v_0^2\, sin(2\theta)}{g}$$

Here we used the following trigonometric identity:

$$2\cos(\theta)\sin(\theta) = \sin(2\theta)$$

We can prove that the path of the projectile is a parabola

$$y(x) = Ax^2 + Bx + C$$

In order to do that we have to pass from time t to position x as follows:

$$y(t) = v_{0y}\, t - \frac{g}{2}t^2$$
$$t = \frac{x}{v_{0x}}$$
$$y(x) = v_{0y}\frac{x}{v_{0x}} - \frac{g}{2}\frac{x^2}{v_{0x}^2} = \tan(\theta)x - \frac{g}{2v_0^2\cos^2(\theta)}x^2$$

Hence

$$A = -\frac{g}{2v_0^2\cos^2(\theta)} \quad B = \tan(\theta) \quad C = 0$$

2.2.2 Circular Motion

Rotating objects are so essential to technology that we barely notice them. Examples include wheels on cars, bicycles, trains, and lawnmowers; the gears and hands of analogue clocks. To describe circular motions we define a set of variables that are analogous to displacement, velocity, and acceleration, but use angular measure instead of linear distance.

$$x(t) \rightleftarrows \varphi(t)$$
$$v = \dot{x}(t) \rightleftarrows \dot{\varphi}(t) = \omega$$
$$a = \ddot{x}(t) \rightleftarrows \ddot{\varphi}(t) = \alpha$$
$$\dot{v} \rightleftarrows \dot{\omega}$$

Instead of displacement, we speak of *angular displacement* $\Delta\varphi$. Definition of angular displacement

$$\Delta\varphi = \varphi_{\text{final}} - \varphi_{\text{initial}}. \tag{2.10}$$

The usual convention is that a positive angular displacement represents a counterclockwise rotation and a negative angular displacement represents clockwise rotation. The

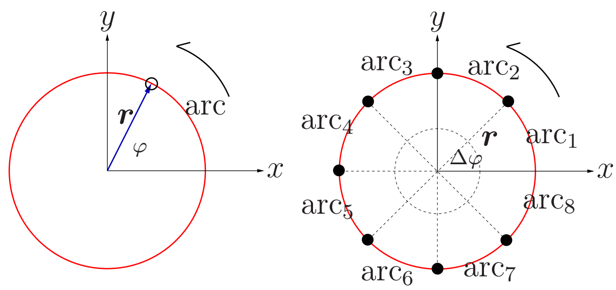


Figure 2.8: On the left: the orientation of the position vector $\boldsymbol{r}$ and the angular coordinate φ. On the right: uniform circular motions.

angular coordinate φ is measured in *radians*. The angle ϕ in radians is defined by arc length s divided by the radius r. For this reason the radian can be considered as a dimensionless quantity. Since "rad" is not a physical unit like meters or kilograms. For this reason, we can drop "rad" whenever there is no chance of being misunderstood.

The ω angular velocity is completely analogous with v. For this reason the definition of angular velocity:

$$\omega(t) = \lim_{\Delta t \to 0} \frac{\varphi(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \langle \omega \rangle = \left. \frac{d\,\varphi(t')}{dt'} \right|_{t'=t} \tag{2.11}$$

The derivative of the angular displacement is often represented by an overdot.

$$\omega = \dot{\varphi}(t)$$

The unit of $[\omega] = 1/\mathrm{s}$.

2.2.3 Uniform Circular Motions

When the speed of a point moving in a circle is constant, its motion is called uniform circular motion (see figure (2.8)). Even though the speed of the point is constant, the velocity is not: the direction of the velocity is changing. The time for the point to travel completely around the circle is called the *period* of the motion, T. The *frequency* of the motion, which is the number of revolution per unit time, is defined as $f = 1/T$.

$$\langle \omega \rangle = \omega = \frac{2\pi}{T} = 2\pi f$$

2.2.4 Analogue Clock

Our analogue clock shows 12 pm. How long does it take until the hands of the analogue clock are at a right angle?

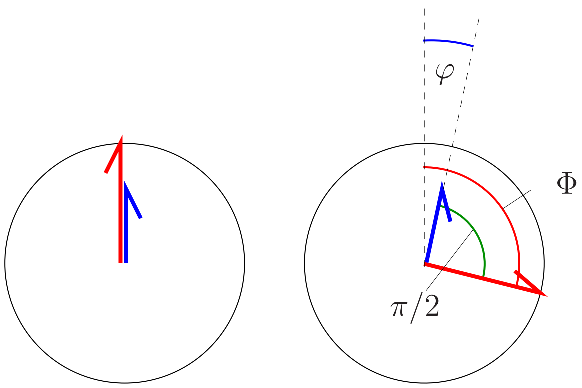


Figure 2.9: An analogue clock shows 12 o'clock.

In this particular case we should use clockwise representation of the angular coordinates. It should be noted that any representation is a matter of convention therefore in some cases we may deviate from the rules. In this case, the deviation from the convention simplifies the consideration.

According to the geometrical criterion

$$\Phi - \varphi = \frac{\pi}{2}$$

the hands of the clock are moving with uniform circular motion and constant angular velocity.

$$\Omega\, t - \omega\, t = \frac{\pi}{2}$$

The time period of the bigger hand is denoted by T_{b} and the time period of the smaller is indicated by T_{s}, hence

$$\frac{2\cancel{\pi}}{T_{\mathrm{b}}}t - \frac{2\cancel{\pi}}{T_{\mathrm{s}}}t = \frac{\cancel{\pi}}{2}$$

$$t = \frac{\frac{1}{2}}{\frac{2}{T_{\mathrm{b}}} - \frac{2}{T_{\mathrm{s}}}} = \frac{\frac{1}{2}}{\frac{2}{60\cdot 60} - \frac{2}{12\cdot 60\cdot 60}}$$

$$t = \frac{1}{\frac{1}{15\cdot 60} - \frac{1}{3\cdot 60\cdot 60}} = \frac{10800}{12-1}$$

$$t = \frac{10800}{11} = 981,8181\,[\mathrm{s}]$$

$$t \approx 16,36\,\mathrm{minutes}$$

2.2.5 $\boldsymbol{v}$ and $\boldsymbol{a}$ during uniform circular motion

The α angular acceleration is completely analogous with a. For this reason the definition of angular acceleration is as follows:

$$\alpha(t) = \lim_{\Delta t \to 0} \frac{\omega(t+\Delta t) - \omega(t)}{\Delta t} = \lim_{\Delta t \to 0} \langle \alpha \rangle = \left.\frac{d\,\omega(t')}{dt'}\right|_{t'=t} \tag{2.12}$$

The derivative of the angular velocity is often represented by overdots.

$$\alpha(t) = \dot{\omega}(t) = \ddot{\varphi}(t)$$

The unit of $[\alpha] = 1/\mathrm{s}^2$.

Up to this point we've dealt exclusively with the Cartesian (or Rectangular, or $x-y$) coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. So, we will start looking at the polar coordinate system. Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system at point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically. This is, however, not the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive x-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point. Unit vectors may be used to represent the axes of a Cartesian coordinate system. For instance, the unit vectors in the direction of the x, y axes of a two dimensional Cartesian coordinate system are

$$\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.13}$$

Two orthogonal unit vectors appropriate to azimuthal symmetry are $\boldsymbol{e}_{\mathrm{r}}$ representing the direction along which the distance of the point from the axis of symmetry is measured (see figure (2.10)). $\boldsymbol{e}_{\varphi}$ representing the direction of the motion that would be observed if the point were rotating counterclockwise about the symmetry axis (see figure (2.10)) . The azimuthal unit vectors can be expressed by the Cartesian unit vectors as follows:

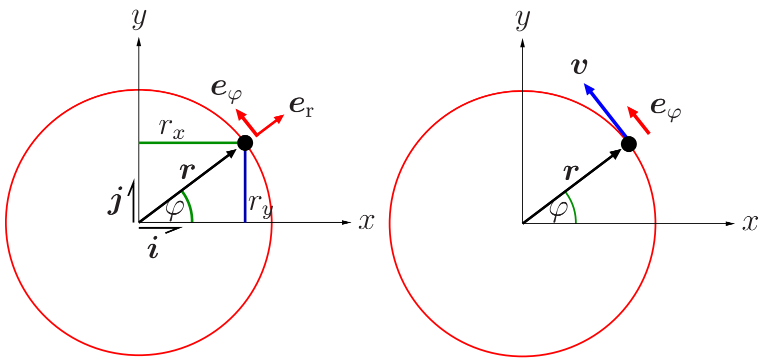


Figure 2.10: Orthogonal unit vectors of the Cartesian and polar coordinate systems.

$$\boldsymbol{e}_{\mathrm{r}} = \cos(\varphi)\,\boldsymbol{i} + \sin(\varphi)\,\boldsymbol{j} \tag{2.14}$$

$$\boldsymbol{e}_{\varphi} = -\sin(\varphi)\,\boldsymbol{i} + \cos(\varphi)\,\boldsymbol{j} \tag{2.15}$$

The rotating $\boldsymbol{r}$ vector takes the following form:

$$\boldsymbol{r} = r\,\boldsymbol{e}_r \tag{2.16}$$

$$\boldsymbol{r} = r\,\{\cos(\varphi(t))\,\boldsymbol{i} + \sin(\varphi(t))\,\boldsymbol{j}\} \tag{2.17}$$

in case of uniform circular motion $\varphi(t) = \omega \cdot t$, hence

$$\boldsymbol{r} = r\,\{\cos(\omega\,t)\,\boldsymbol{i} + \sin(\omega\,t)\,\boldsymbol{j}\} \tag{2.18}$$

By definition the $\boldsymbol{v} = \dot{\boldsymbol{r}}$, hence

$$\boldsymbol{v} = r\dot{\boldsymbol{e}}_r = r\omega\,\underbrace{\{-\sin(\omega\,t)\,\boldsymbol{i} + \cos(\omega)\,\boldsymbol{j}\}}_{\boldsymbol{e}_{\varphi}}$$

$$\boldsymbol{v} = r\omega\,\boldsymbol{e}_{\varphi} \tag{2.19}$$

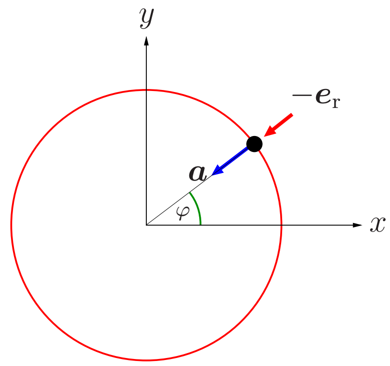


Figure 2.11: The orientation of the acceleration vector of the uniform circular motion.

Equation 2.19 relates the linear and angular speeds. By definition the $\boldsymbol{a} = \dot{\boldsymbol{v}}$, hence

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = r\omega\dot{\boldsymbol{e}}_t = r\omega^2 \underbrace{\{-\cos(\omega\, t)\, \boldsymbol{i} - \sin(\omega)\, \boldsymbol{j}\}}_{(-\boldsymbol{e}_r)}$$

$$\boldsymbol{v} = -r\omega^2\, \boldsymbol{e}_r$$

The centripetal (radial) acceleration of the points of a rotating particle is

$$\boldsymbol{a}_r = -r\omega^2\, \boldsymbol{e}_r \tag{2.20}$$

There is a minus sign in this formula because the vectors $\boldsymbol{r}$ and $\boldsymbol{a}_r$ have opposite direction.

2.2.6 Kinematics of the Simple Harmonic Motion (SHM)

Vibration, one of the most common kinds of motion, is repeated motion back and forth along the same path (see figure (2.12)). To find the mathematical description of the kinematics of SHM, we analyse the uniform circular motion of a point particle. The particle is moving counterclockwise around a circle of radius A at a constant angular velocity ω. The location of the particle at any time is then given by the angle θ

$$\theta(t) = \omega t$$

The motion of the particle's shadow has the same x-component as the particle itself. We find that

$$x(t) = A\cos(\theta) = A\cos(\omega t)$$

We know that the velocity vector is always tangential and the magnitude of it $R\omega$, so:

$$v_x = -R\omega\sin(\theta) = -A\omega\sin(\omega t)$$

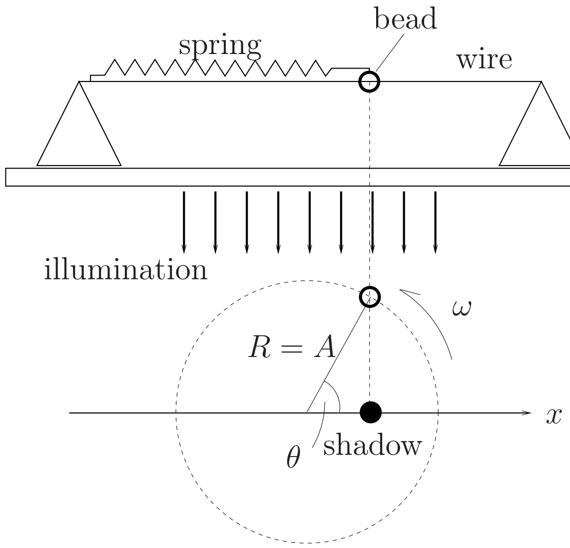


Figure 2.12: The basic relationship between uniform circular motion and simple harmonic motion.

Since the particle moves in uniform circular motion, its acceleration is constant in magnitude but not in direction; the acceleration is toward the centre of the circle. The magnitude of the radial acceleration is shown to be

$$a = A\omega^2$$

At any instant the direction of the acceleration vector is opposite to the direction of the displacement vector. Therefore,

$$a_x(t) = -a\cos(\theta) = -\omega^2 A\cos(\omega t)$$

Because $x(t) = A\cos(\omega t)$ the acceleration can be expressed as the function of the position

$$a(x) = \omega^2 x(t)$$

2.2.7 Uniform motion along a parabolic wire

Given an object with mass m traveling along a parabolic wire ($y(x) = bx^2$) with constant speed v. The orientation of the parabolic wire can be seen in figure (2.14). What are the mathematical forms of vectors $\boldsymbol{v}$ and $\boldsymbol{a}$?

We have introduced a tangential unit vector $\boldsymbol{e}_\tau$.

$$\boldsymbol{v} = v \cdot \boldsymbol{e}_\tau$$
$$\tan\varphi = \frac{d}{dx}\left(bx^2\right) = 2bx$$
$$\varphi = \tan^{-1}(2bx)$$

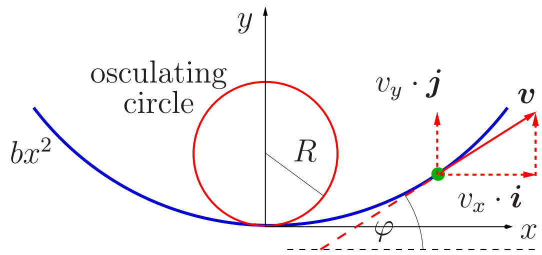


Figure 2.13: Uniform motion along a parabola.

$$\boldsymbol{v} = v\cos(\varphi)\cdot\boldsymbol{i} + v\sin(\varphi)\cdot\boldsymbol{j}$$
$$\boldsymbol{v} = v\cos\left\{\tan^{-1}(2bx)\right\}\cdot\boldsymbol{i} + v\sin\left\{\tan^{-1}(2bx)\right\}\cdot\boldsymbol{j}$$

We can use the following trigonometric identities:

$$\cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}$$
$$\sin\left(\tan^{-1}x\right) = \frac{x}{\sqrt{1+x^2}}$$

$$\boldsymbol{v} = \frac{v}{\left(1+4b^2x^2\right)^2}\cdot\boldsymbol{i} + \frac{2bxv}{\left(1+4b^2x^2\right)^2}\cdot\boldsymbol{j}$$

We can check this formula because we know that in the position $x = 0$, the particle moves parallel with the x axis hence v_y must be vanished.

$$\boldsymbol{v} = \frac{v}{\left(1+4b^2\,0^2\right)^2}\cdot\boldsymbol{i} + \frac{2b\,0\,v}{\left(1+4b^2\,0^2\right)^2}\cdot\boldsymbol{j} = v\cdot\boldsymbol{i}$$

The acceleration vector can be obtained by the derivative of the velocity vector with respect to time

$$\boldsymbol{a} = -\frac{4b^2v^2}{\left(1+4b^2x^2\right)^2}x\cdot\boldsymbol{i} + \frac{2b\,v^2}{\left(1+4b^2x^2\right)^2}\cdot\boldsymbol{j}$$

We know that the tangential component of the acceleration must be zero in $x = 0$ and also have to be vertically directed, hence

$$\boldsymbol{a} = -\frac{4b^2v^2}{\left(1+4b^2\,0^2\right)^2}0\cdot\boldsymbol{i} + \frac{2bv^2}{\left(1+4b^2\,0^2\right)^2}\cdot\boldsymbol{j} = 2bv^2\cdot\boldsymbol{j}$$

This result can be obtained by using the osculating circle of the parabola in the apex. The radius of curvature of the osculating circle is

$$\frac{1}{R} = \left.\frac{y''}{\left(1+y'^2\right)^{3/2}}\right|_{x=0} = \left.\frac{2b}{\left(1+(2bx)^2\right)^{3/2}}\right|_{x=0} = 2b$$

In position $x = 0$ there is an instant uniform circular motion, hence

$$a = \frac{v^2}{R} = 2bv^2$$

The acceleration pointed to the centre of the osculating circle, so it is parallel with the vertical axis.

2.2.8 General Kinematics in a Plane

Without any detailed explanation, we give the general expressions for two-dimensional motion of a particle in polar coordinate system

$$\boldsymbol{r}(t) = \rho(t)\boldsymbol{e}_\rho(t) \tag{2.21}$$
$$\boldsymbol{v}(t) = \dot{\rho}(t)\boldsymbol{e}_\rho(t) + \rho(t)\,\dot{\varphi}\,\boldsymbol{e}_\varphi(t) \tag{2.22}$$
$$\boldsymbol{a}(t) = \left(\ddot{\rho} - \rho\,\dot{\varphi}^2\right)\boldsymbol{e}_\rho(t) + [2\dot{\rho}\,\dot{\varphi} + \rho\,\ddot{\varphi}]\,\boldsymbol{e}_\varphi(t) \tag{2.23}$$

where $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\varphi$ are orthogonal unit vectors of the plane polar coordinate system. Unlike stationary unit vectors of the Cartesian coordinate system ($\boldsymbol{i}$ and $\boldsymbol{j}$) the orientation of $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\varphi$ is continuously changing as time elapses.

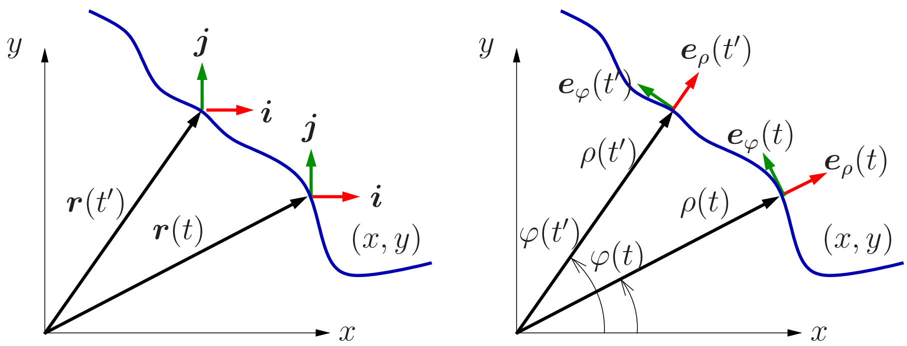


Figure 2.14: Orientation of the position and orthogonal unit vectors during general two-dimensional motion.

Let us consider a spiral motion. We assume that the particle moves away from the origin with constant speed v_0 and the angular velocity is also constant ω_0. For this reason both ρ and φ grows linearly with time

$$\rho(t) = v_0 t \tag{2.24}$$
$$\varphi(t) = \omega_0 t \tag{2.25}$$

The velocity and acceleration of the particle can be expressed as follows

$$\boldsymbol{v}(t) = v_0\boldsymbol{e}_\rho + v_0\,t\,\omega_0\,\boldsymbol{e}_\varphi \tag{2.26}$$
$$\boldsymbol{a}(t) = -v_0\,t\,\omega_0^2\,\boldsymbol{e}_\rho + 2v_0\,\omega_0\boldsymbol{e}_\varphi \tag{2.27}$$

In case of circular motion $\rho = R = \text{const}$ hence $\dot{\rho} = 0$ and $\ddot{\rho} = 0$.

$$\boldsymbol{r}(t) = R\boldsymbol{e}_\rho \tag{2.28}$$
$$\boldsymbol{v}(t) = R\dot{\varphi}\,\boldsymbol{e}_\varphi \tag{2.29}$$
$$\boldsymbol{a}(t) = -R\,\dot{\varphi}^2\boldsymbol{e}_\rho + R\,\ddot{\varphi}\,\boldsymbol{e}_\varphi \tag{2.30}$$

where $-R\,\dot{\varphi}^2\boldsymbol{e}_\rho$ is the centripetal and $R\,\ddot{\varphi}\,\boldsymbol{e}_\varphi$ is the tangential acceleration.

Chapter 3

Dynamics

Kinematics describes the motion of bodies without being concerned with why a body moves exactly in a given way, and not in a different one. Dynamics studies the motion of bodies in connection with its causes resulting in the occurrence of a special kind of motion.

3.1 Principia

In 1687, Isaac Newton published one of the greatest scientific works of all time, his *Philosophiae Naturalis Principia Mathematica (or Principia for short).* The Latin title translates as *The Mathematical Principles of Natural Philosophy*. In the Principia, Newton stated three laws of motion that form the basis of classical mechanics. Together with his law of universal gravitation, Newton's laws showed for the first time that the motion of the heavenly bodies (the Sun, the planets, and their satellites) and the motion of earthly bodies can be understood using the same physical laws. Newton's laws were the result of generalizing a great amount of experimental facts. *Their correctness is confirmed by the agreement of the corollaries following from them with experimental results.* For this reason the axioms of classical mechanics are different from the axioms of mathematics. Newtonian mechanics achieved such great successes during two centuries that many physicists of the 19th century were convinced of its perfection. It was considered that the explanation of any physical phenomenon required its reduction to a mechanical process obeying Newton's laws. With the development of science, however, new facts were uncovered for which no place could be found within the confines of classical mechanics. These facts were explained in new theories—theory of relativity and quantum mechanics.

It is worth emphasizing that the development of science has not eliminated classical mechanics, but has only shown its limited applicability.

3.2 Newton's First Law (Law of inertia)

Aristotle had formulated his view that the natural state of an object is rest; and, for an object to remain in motion, a force would have to act upon it continuously. Galileo conjectured that, in the absence of friction and other resistive forces, no continued force is needed to keep an object moving. However, Galileo thought that the sustained motion of an object would be in a great circle around Earth. Descartes claimed that the motion of an object free of any forces should be along a straight line rather than a circle.

Newton's first law is formulated as follows: *every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by external forces to change that state.* A reference frame in which Newton's first law is obeyed is called an inertial one. The law itself is often called the *law of inertia.* A reference frame in which Newton's first law is not obeyed is called a non-inertial reference frame. There is an infinite multitude of inertial reference frames. Any reference frame moving uniformly in a straight line relative to an inertial frame will be also an inertial one.

The Earth moves relative to the Sun and the stars along a curvilinear trajectory having the shape of an ellipse. Curvilinear motion always occurs with a certain acceleration. The Earth also rotates about its axis. For this reason, the Earth's surface is not inertial. The acceleration of such a frame, however, is so small that it may be considered practically inertial in a great number of cases. But sometimes the non-inertial nature of the reference frame associated with the Earth is observable (Fouceault's pendulum).

Every body resists attempts to change its state of motion. This property of bodies is called *inertia.* It is characterised quantitatively by a physical quantity called the *mass* of the body.

3.3 Newton's Second Law

Newton's second law states that *the rate of change of the momentum of a body equals the force* $\boldsymbol{F}$ *acting on the body*:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} \tag{3.1}$$

substituting $m\boldsymbol{v}$ for $\boldsymbol{p}$ we get

$$\frac{d\,(m\boldsymbol{v})}{dt} = \dot{m}\boldsymbol{v} + m\,\dot{\boldsymbol{v}} \tag{3.2}$$

We know from kinematics that $\dot{\boldsymbol{v}} = \boldsymbol{a}$ Hence,

$$\frac{d\,(m\boldsymbol{v})}{dt} = \dot{m}\boldsymbol{v} + m\,\boldsymbol{a} \tag{3.3}$$

There are several problems in the framework of classical mechanics where the mass changes during the process: avalanche, a falling raindrop, the motion of a rocket, the motion of leaking oscillator, the motion of the tip of a whip. If the mass is assumed to be constant $\dot{m} = 0$ we can write the equation (3.3) in the familiar form

$$\boldsymbol{F} = m\,\boldsymbol{a} \tag{3.4}$$

Equation has called forth and is continuing to call forth many controversies among physicist. To date, there is no generally adopted interpretation of this relation. The complication consists in that there are no independent ways to determining the quantities m and F. First of all, let us deal with a common mental mistake made by beginners. It is perhaps tempting to read Newton's second law as a literal identification of force with mass times acceleration. This is not the way to think of it. The acceleration is a property of a given body's state (of motion) and is only defined on a given curve. The mass is an intrinsic property of the body. The force is an external field quantity characterizing the environment and the body's interaction with it. *Don't think of $F = ma$ as an identification of two quantities.* Think of it as expressing the response (acceleration) of a body to an external influence (force). *The right hand side of $F = ma$ refers to the body; the left hand side refers to its environment. It mathematically states the cause and effect relationship between force and changes in motion.*

We must underline the fact that Newton's second law is an experimental one. It took shape as a result of generalization of the data of experiments and observations.

One question immediately arises: "What do we mean by a force?" We define a *force as the push or pull exerted by one body on another body.* This definition is not quantitative but emphasizes the fact that we are entitled to speak of a "force" *only* when we can identify the body which is exerting the force and the body on which the force is being exerted.

3.3.1 The Superposition of Forces

There is a tacit assumption we make when using Newton's laws that really should get a law all its own. It is often called as the superposition principle. This is just the postulate, that when two or more forces act on a particle, the resulting net force is just the vector sum of the individual force vectors:

$$\boldsymbol{F}_{\text{total}} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots + \boldsymbol{F}_n = \sum_{n=1}^{N} \boldsymbol{F}_n \tag{3.5}$$

It need not be true that forces combine as vectors, or even that forces combine in a linear fashion. In fact, there are situations where the superposition principle is not valid. For example, strong gravitational fields don't behave this way. As usual, for a wide range of physical situations the superposition principle does hold with extremely good accuracy, so we take it as given.

3.3.2 Inertial Frames

Now that we know what we mean by a force, we can ask: "With respect to which axes is it true that a particle subject to no forces moves with constant velocity?" A set of axes is frequently called a "frame of reference", and those axes with respect to which Newton's first law is true are called *inertial frames.*

An inertial reference frame is axes which are non-rotating with respect to the distant stars and whose origin moves with uniformly along a straight line. For most purposes, axes attached to the earth appear to be in inertial frame.

However, while a reference frame (the train) is accelerating, the passenger has the feeling that something is pushing her back into her seat. Nevertheless, we do not acknowledge that there is any force pushing the woman toward the back of the car since we cannot point to any piece of matter which is exerting such a force on the passenger. In this case the train is considered as a non-inertial reference frame.

3.3.3 The Equation of Motion

The relationship between the particle's motion, $\boldsymbol{r} = \boldsymbol{r}(t)$, and the applied force can be viewed as a differential equation. Mathematically we present the force as some given function of position, velocity and explicitly of time, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v}, t)$. We get in general a coupled system of 3 second-order, ordinary differential equations. These are called the *equations of motion.*

$$m\ddot{\boldsymbol{r}} - \boldsymbol{F}\left(\boldsymbol{r}(t), \boldsymbol{v}(t), t\right) = 0 \tag{3.6}$$

Solving of the set of this kind of equations is among the most difficult problems in mathematics. One of the most famous problems in classical mechanics is the so called three-body problem. The three-body problem is the problem of taking an initial set of data that specifies the positions, masses, and velocities of three bodies for some particular point in time and then determining the motions of the three bodies, in accordance with Newton's laws of motion and of universal gravitation, which are the laws of classical mechanics. Unlike two-body problems, there is no general closed-form solution for every condition, and numerical methods are needed to solve these problems. The mathematician Henry Poincaré showed that there is no general analytical solution for the three-body problem given by algebraic expressions and integrals. The motion of three bodies is generally non-repeating, except in special cases.

However, the mathematicians have pointed out that some kind of solutions always exist and the solutions are not unique in general. There are many solutions. This is good because, given a force field, we know that infinitely many motions are possible. Finally, the solutions are uniquely determined by their initial position and initial velocity. This is a strong physical prediction of Newtonian mechanics, one which is readily born out by experience and experiment.

3.3.4 Formal Solution of the Equation of Motion

In this case, the force being given by $F = F(t)$ implies that it is an explicit function of time; hence Newton's second law may be written as

$$m\frac{dv}{dt} = F(t) \tag{3.7}$$

which on integration gives, assuming that $v = v_0$ at $t = t_0$,

$$v = v_0 + \frac{1}{m}\int_{t_0}^{t} F(t)dt \tag{3.8}$$

$$\frac{dx(t)}{dt} = v_0 + \frac{1}{m}\int_{t_0}^{t} F(t)dt \tag{3.9}$$

Integrating again

$$x = x_0 + v_0\left(t - t_0\right) + \int_{t_0}^{t}\left(\frac{1}{m}\int_{t_0}^{t} F(t)dt\right)dt \tag{3.10}$$

Since there are two integrations, we may use two variables t' and t'' and write equations (3.11) as

$$x = x_0 + v_0\left(t - t_0\right) + \frac{1}{m}\int_{t_0}^{t} dt' \int_{t_0}^{t'} F(t'')dt'' \tag{3.11}$$

However this integration usually can not be performed (with some exceptions). Hence most of the practical problems are treated differently.

3.3.5 Constant Applied Force

We are interested in studying the motion of a particle when the applied force acting on the particle is constant in time. Since F is constant, so will be the acceleration a, and we may write Newton's second law as

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{F}{m} = a = \text{constant} \tag{3.12}$$

The equation may be solved by direct integration provided we know the initial conditions. Solving the equation (3.12) gives us a familiar result obtained in elementary mechanics, as we will show now. Let us assume that at $t = 0$, the initial velocity, and at time t the velocity is v_0. Thus, from the equation (3.12),

$$\int_{v_0}^{v} dv = \int_{0}^{t} adt \tag{3.13}$$

which on integration yields

$$v = v_0 + at \tag{3.14}$$

Substituting $v = dx/dt$ in the equation (3.13) and again assuming the initial condition that $x = x_0$ at $t = 0$, we get by direct integration

$$x = x_0 + v_0 t + \frac{a}{2}t^2 \tag{3.15}$$

By eliminating t between equations (3.12) and (3.13), we get

$$v^2 = v_0^2 + 2a(x - x_0) \tag{3.16}$$

Equations (3.12), (3.13), and (3.16) are the familiar equations that describe the translational motion of a particle in one dimension.

One of the most familiar examples of motion with constant force is the motion of freely falling bodies. In this case, a is replaced by g, the acceleration due to gravity, having the value $g \approx 9.81$ m/s^2.

3.4 Newton's Third Law

Any action of bodies on one another has the nature of mutual interaction: if body 1 acts on body 2 with the force $\boldsymbol{F}_{21}$, then the body 2, in turn, acts on body 1 with force $\boldsymbol{F}_{12}$. Newton's third law states that *the forces exerted by interacting bodies on each other are equal in magnitude and opposite in direction*. Using the former notation for such forces, the third law can be expressed in the form of the following equation:

$$\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}. \tag{3.17}$$

It follows from Newton's third law that forces appear in pairs: for any force applied to a body there is another force equal in magnitude and opposite in direction applied to the second body interacting with the first one.[1]

3.5 Forces

Four kinds of fundamental interactions are distinguished in physics *1) gravitational*, *2) electromagnetic*, *3) strong or nuclear* (ensuring the binding of particles in an atomic nucleus), and *4) weak* interaction (responsible for many processes of elementary particle decay).

Within the confines of classical mechanics, we have to do with gravitational and electromagnetic forces, and also with elastic and friction forces. The forces of interaction between molecules have an electromagnetic origin. Consequently, elastic and friction forces are electromagnetic in their nature.

Gravitational and electromagnetic forces are fundamental up to now they cannot be reduced to other simpler forces. Elastic and friction forces, on the other hand, are not fundamental.

[1] Newton's third law is not always correct. As an example of the violation of Newton's third law, we can take a system of two charged particles q_1 and q_2 moving at the given moment. It can be proved—in electrodynamics—that the magnetic forces acting on the particles are not oppositely directed.

3.5.1 Gravitational, Electric and Magnetic Forces

Let us consider a system of two electrically neutrally particles m_1 and m_2 at the distance r from each other. Owing to universal gravitation, these particles attract each other with the force

$$F = G\frac{m_1\, m_2}{r^2}, \tag{3.18}$$

where $G = 6.674 \cdot 10^{-11}$ Nm^2/kg^2 is the universal gravitational constant.

The magnitude of the force with which two point charges q_1 and q_2 at rest interact is determined by Coulomb's law:

$$F = k\frac{q_1\, q_2}{r^2}, \tag{3.19}$$

where k is a constant of proportionality depending on the units chosen. In SI system $k = 9 \cdot 10^9$ m/F.

If the charges are moving, then magnetic forces act on them in addition to the electric force in the presence of external magnetic field. The magnetic force acting on a point charge q moving with the velocity $\boldsymbol{v}$ in a magnetic field of magnetic flux density $\boldsymbol{B}$ is determined by

$$\boldsymbol{F} = q\,[\boldsymbol{v} \times \boldsymbol{B}]\,. \tag{3.20}$$

3.5.2 Elastic Forces

Any real body becomes deformed (changes its dimensions and shape), under the action of forces applied to it. If the body regains its initial dimensions and shape after the action of the forces stops, the deformation of strain is called elastic. Elastic deformation is observed when the force producing the deformation does not exceed a definite limit, called the *elastic limit*, for each concrete body.

Let us consider an ideal massless spring which has equilibrium length L. The spring is horizontally oriented and the left end of the spring is attached to a wall. At a given moment a force is exerted on the right end of the spring. Let the actual length of the spring at a given instant be $L + x$. It is assumed that in equilibrium the coils of the spring are partially open so that x can be positive or negative. If x is positive the spring exerts a force to the left and, if x is negative, the spring exerts a force to the right. An "ideal" spring is one which obeys Hooke's Law, which says that the magnitude of the force is proportional to the magnitude of x. The quantitative statement of Hooke's Law is

$$F(x) = -kx \tag{3.21}$$

where k is a constant of proportionality called the *spring constant*. The minus sign ensures that if x is positive (negative) the force is directed to the left (right). Hooke's Law is not a fundamental law of nature, but most springs obey Hooke's Law if x is small enough. Every spring will deviate from Hooke's Law if it is stretched or compressed too far.

Homogeneous bars behave in tension or uniaxial compression like a spring. The statement that the elastic force and the deformation are proportional to each other is also called Hook's law.

$$\sigma = E\varepsilon \tag{3.22}$$

where $\sigma = F/A$ is the normal stress, $\varepsilon = \Delta l/l_0$ the relative elongation (strain) and E is called the modulus of elasticity or Young's modulus. It is measured in pascals ($\mathrm{N/m^2}$).

3.5.3 Friction Forces

Forces of friction appear when contacting bodies or their parts move relative to each other. The friction occurring in the relative movement of two contacting bodies is called *external*; the friction between the molecules of the fluid is called *internal*. Friction between the surfaces of two solids in the absence of any intermediate layer, for instance, a lubricant between them, is called *dry*. Friction between a solid and a fluid, and also between the layers of a fluid, is called *viscous*.

Two kinds of friction are distinguished: *sliding* and *rolling*. Forces of friction are directed along a tangent to the surfaces (or layers) in contact so that they resist the relative displacement of these surfaces. In dry friction, a force of friction appears not only when one surface slides over another one, but also when attempts are made to set up such sliding motion. As it can be seen in Fig.(3.1) the static friction acts to

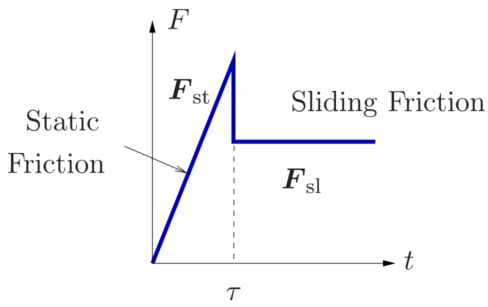


Figure 3.1: The magnitude of the static friction can vary. Reaching a critical value turns into sliding friction.

prevent objects from starting to slide. As time goes by the magnitude of the static friction is increasing up to a critical value. At time moment $t = \tau$ the body begins to move and the magnitude of the sliding friction force drops significantly. The force of sliding friction slightly depends on the relative speed, but this dependence can be neglected. The maximum force of static friction, and also the force of sliding friction do not depend on the area of contact between bodies and are approximately proportional to the magnitude of the normal force pressing the contacting surface together.

$$F = \mu_{\mathrm{s}} F_n \quad \text{static,} \tag{3.23}$$

$$\boldsymbol{F} = -\mu_{\mathrm{k}} F_n \frac{\boldsymbol{v}}{v} \quad \text{sliding.} \tag{3.24}$$

The dimensionless proportionality constant μ is called the *coefficient of friction*. It is commonly thought (from daily life) that the static coefficients of friction are higher than the dynamic or kinetic values $\mu_\mathrm{s} > \mu_\mathrm{k}$.[2] It depends on the nature and state of the contacting surfaces, on their roughness and on their contamination.

Friction forces play a very great part in nature. Friction is often a great help to us in our everyday life. Nonetheless, in case of friction in bearing or between the hubs of a wheel measures have to be taken to reduce it as much as possible. The most radical way of reducing forces of friction is to replace sliding friction with rolling friction.

Rolling resistance is the force that resists the rolling of a wheel or other circular object along a surface caused by deformations in the object or surface. Generally the force of rolling resistance is less than that associated with kinetic friction.

$$F = C_{rr}N$$

where N is the normal force, C_{rr} is the force needed to push (or tow) a wheeled vehicle forward (at constant speed on a level surface, or zero grade, with zero air resistance) per unit force of weight. If railroad steel wheels are rolling on steel rails the rolling resistance coefficient is $C_{rr} = 0.001$.

3.5.4 Constraints

From a dynamical point of view any material system can be regarded as a collection of material particles. The relationship between the quantities determining the position and the velocity of the system of particles is referred to as a *constraints*. These relationships must hold regardless of the initial conditions and the forces acting on the system. A constraint is a geometric kinematic condition that restricts the motion of a body. The constraints are usually given as equality and/or inequality. Here are a few examples of geometric constraints for a particle: the body is constrained to move on a certain curve, on a certain surface, or inside a given volume.

A constraint can be assimilated with a constraint force. The forces of constraint determine the body to move on a certain curve, a certain surface, or in a certain volume. A constraint of a system is a property of the external environment which reduces the degrees of freedom of that system.

3.6 Applications of Newton's Laws

To compile an equation of motion, we must first of all establish what forces act on the body being considered. It is necessary to determine the action of other bodies on the given one that must be taken into account. For example, for a body sliding down an inclined plane, the action exercised by the Earth is important. The gravitational force is characterized by the force mg. Also important the action exercised by the plane which is characterised by the force of the reaction.

[2] With many brake materials the static coefficient of friction is often significantly lower than the dynamic value. It can be as low as 40 - 50% of the dynamic value.

Constraint	Function	Idealization
rope	pulling distant bodies only exerts pulling force	massless undeformable
rod	pushing distant bodies it can exert pulling and pushing	massless undeformable
pulley	rotation of the line of action, not able changes the forces	massless undeformable
surface	block the normal movement exerts only normal force	frictionless
hinge	connects two solid objects allows rotation about a fixed axis	frictionless

Table 3.1: The most commonly used constraints in mechanics.

3.6.1 Movable and Fixed Pulley

Let us consider a mechanical system consist of two blocks are connected by a light weight, flexible cord that passes over a fixed and a moving pulley (see figure (3.2)). The centres of both pulleys are connected to a massless rigid rod. The friction is negligible. Let us describe the motion of the system.

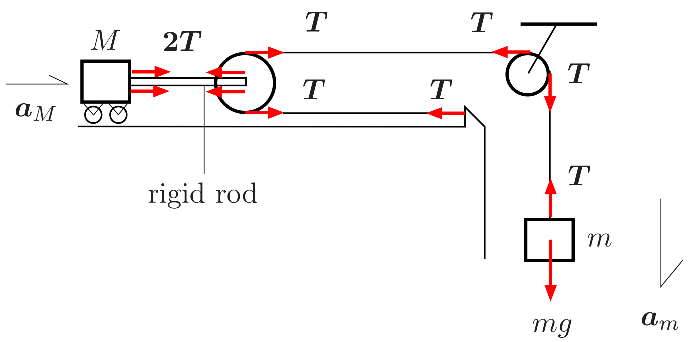


Figure 3.2: Mechanical system behaves as a simple machine.

i) We start by drawing a separate free-body diagram for each block.
ii) Using the free-body diagrams, we write Newton's second law for each block. We get a system of linear equations.
iii) Determine the constraint relationship. It generally reduces the number of unknowns.
iv) Each equation of motion must contain new information about the variables. *We must have at least as many equations as unknowns.*
v) Solve the system of equations.
vi) Discussion: are the results reasonable?

Solving an equation means using algebraic operations to isolate one variable. Many students tend to substitute numerical values into an equation as soon as possible. In many cases that is a mistake. On the one hand symbolic algebra is much easier to follow than a series of numerical calculations. Plugging numbers tends to obscure the logic behind your solution. On the other hand symbolic algebra lets you draw conclusions about how one quantity depends on another. Simultaneous equations are a set of N equations with N unknown quantities.

In case of massless rope, rod or pulley they must be in mechanical equilibrium. This means that the net force acting on them must be zero. For this reason we do not apply equation of motion for such objects, even though they are accelerated. A rope, or a rod is always under tension.

According to Newton's second law and taking into account the notation of Fig.(3.2)

$$mg - T = m\,a_m \tag{1}$$

$$2T = M\,a_M \tag{2}$$

We have two equations and three unknown quantities (T, a_m, and a_M). We have an additional constraint equation as follows:

$$a_M = \frac{a_m}{2} \tag{3}$$

The origin of the constraint relation is based on the fact that if we exert T' force on the vertical rope the machine exerts twice as much force ($2T'$) on the horizontal object. A simple machine uses a single applied force to do work against a single load force. Ignoring friction losses, the work done on the load is equal to the work done by the applied force. The machine can increase the amount of the output force, at the cost of a proportional decrease in the distance moved by the load.

$$\underbrace{F}_{\text{initial}}\ \Delta x = \underbrace{NF}_{\text{output}}\ \frac{\Delta x}{N} \tag{3.25}$$

The ratio of the output to the applied force is called the *mechanical advantage*. Using our simple machine as an example, if the fource is doubled, this leads to a 50 % decrease in the horizontal displacement. Finally, we obtain two independent equations

$$mg - T = m\,a_m \tag{4}$$
$$2\,T = M\,\frac{a_m}{2} \tag{5}$$

After multiplying both sides of the first equation by 2 and adding the two equations we eliminate T and get one equation with one unknown quantity.

$$2\,mg - \cancel{2T} = 2\,m\,a_m \tag{6}$$
$$\cancel{2T} = M\,\frac{a_m}{2} \tag{7}$$

$$2\,mg = \left(2\,m + M\,\frac{1}{2}\right) a_m \tag{8}$$

Hence, the a_m acceleration of the suspended body is

$$a_m = \frac{2mg}{2\,m + M/2} = \frac{4mg}{4\,m + M}$$

From equation (5) we get the tension forces T exerted by the rope sections as follows:

$$T = M\,\frac{a_m}{4} = \left(\frac{M}{\cancel{4}}\right)\frac{\cancel{4}mg}{4\,m + M} = \frac{Mmg}{4\,m + M}$$

It is useful to check whether we get reasonable results. If $M \approx 0$ the suspended body is falling freely, for this reason a_m must be g and the tension force (T) becomes zero.

$$\lim_{M\to 0} a_m = \frac{4mg}{4\,m + 0} = g\checkmark$$
$$\lim_{M\to 0} T = \frac{0 \cdot mg}{4\,m + 0} = 0\checkmark \tag{3.26}$$

3.6.2 Fixed Inclined Plane

A block of mass m is held motionless on a frictionless plane with an angle of inclination θ. The inclined plane is fixed to the ground (see figure (3.3)). The block is released. What is the acceleration of the block? In the example we are considering, it is good to resolve the force of the gravitation force mg into two components—the normal force $mg\cos(\alpha)$ and the tangential force $mg\sin(\alpha)$. Having determined the forces acting on a body, we write an equation of Newton's second law. In our example it has the form

$$m\boldsymbol{a} = m\boldsymbol{g} + \boldsymbol{F}_n + \boldsymbol{F}_\mathrm{f}$$

To perform calculations, we must pass over from vectors to their projections onto the correspondingly chosen direction. Let us project the vectors onto the direction x and

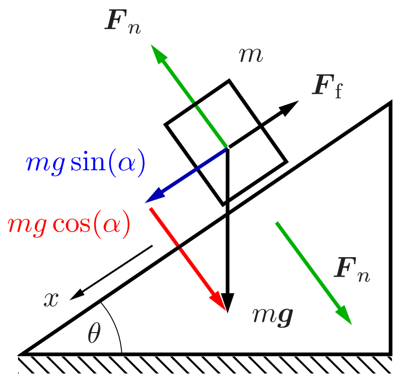


Figure 3.3: A fixed inclined plane and a block.

y shown in figure (3.3). The projections of the vectors are $a_x = a$, $mg_x = mg\sin(\alpha)$, and $F_n = mg\cos(\alpha)$. Since the body does not accelerate along the y direction the net force must be zero, hence

$$F_n - mg\cos(\theta) = 0 \tag{3.27}$$
$$F_n = mg\cos(\theta) \tag{3.28}$$

Consequently, we arrive at the equation

$$\begin{aligned} mg\sin(\alpha) - \mu\, F_n &= m\, a & (1)\\ \cancel{m}g\sin(\alpha) - \mu\, \cancel{m}g\cos(\alpha) &= \cancel{m}\, a \\ a = g\sin(\alpha) - \mu\, g\cos(\alpha) & \end{aligned}$$

3.6.3 Moving Inclined Plane

A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. As it can be seen in figure (3.4) the plane rests on a frictionless horizontal surface so it is not fixed anymore. The block is released. What is the horizontal acceleration of the inclined plane? Instead of the normal force F_n we will use a simplified notation N. It is useful to introduce vertical and horizontal components of the forces and the accelerations as well.

$$mg - N_y = m\, a_{0y} \tag{1}$$
$$-N_x = m\,(a_M - a_{0x}) \tag{2}$$
$$N_x = M\, a_M \tag{3}$$

The former equations can be written in a more detailed form

$$mg - N\cos(\theta) = m\, a_0\sin(\theta) \tag{1}$$
$$-N\sin(\theta) = m\,(a_M - a_0\cos(\theta)) \tag{2}$$
$$N\sin(\theta) = M\, a_M \tag{3}$$

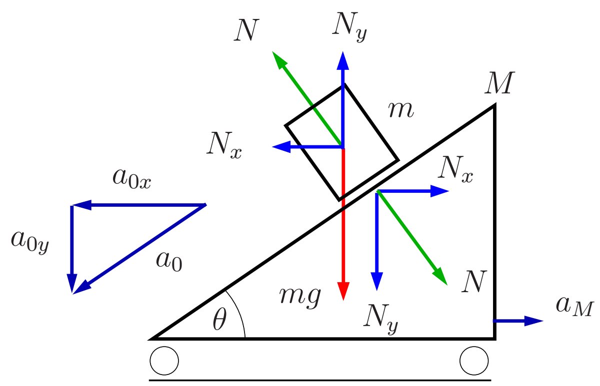


Figure 3.4: Freely moving inclined plane and a block.

It is useful to multiply the first equation by $\sin(\theta)$ and multiply the second one by $\cos(\theta)$. Adding up these equations we can eliminate the normal force N. The solutions of the system of equations are the following:

$$a_0 = \frac{g\sin(\theta)\,(m+M)}{m\sin(\theta)^2 + M},$$

$$a_M = \frac{mg\cos(\theta)\sin(\theta)}{m\sin(\theta)^2 + M},$$

$$N = \frac{Mmg\cos(\theta)}{m\sin(\theta)^2 + M}.$$

We can check whether our results are reasonable or not. In the special case if M is much more massive compared to m the incline is practically fixed ($a_M = 0$), hence the acceleration of m is $g\sin(\theta)$ and the normal force exerted by the incline is $mg\cos(\theta)$.

$$a_0 = \lim_{M\longrightarrow\infty} \frac{g\sin(\theta)\left(\frac{m}{M}+1\right)}{\frac{m}{M}\sin(\theta)^2 + 1} = g\sin(\theta)\checkmark$$

$$a_M = \lim_{M\longrightarrow\infty} \frac{mg\cos(\theta)\sin(\theta)}{m\sin(\theta)^2 + M} = 0\checkmark$$

$$N = \lim_{M\longrightarrow\infty} \frac{mg\cos(\theta)}{\frac{m}{M}\sin(\theta)^2 + 1} = mg\cos(\theta)\checkmark$$

3.6.4 Sliding on a Non-ideal Surface

According to figure (3.5) a massive block is moving at initial speed (v_0) in an ideal horizontal surface until it reaches a fixed platform. The surface of the platform is not ideal, it can be characterized by μ. How far does the block go?

According to Newton's second law only the sliding friction force will change the

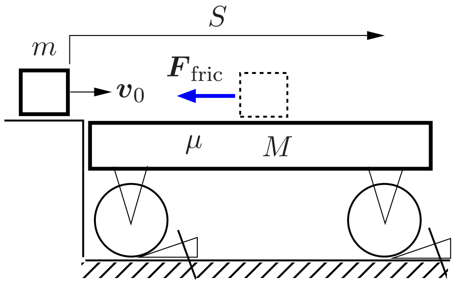


Figure 3.5: Slowing down on a fixed surface.

motion of the block.

$$\begin{aligned} F_{\text{fric}} &= ma \\ \mu F_n &= ma \\ \mu \cancel{m} g &= \cancel{m} a \\ \mu\, g &= a \end{aligned}$$

The direction of motion and the acceleration is oppositely directed, hence $a < 0$, hence $\mu\, g < 0$ (deceleration).

The relation $a = \mu g$ shows that the block slows down with a constant acceleration, hence one of our kinematic formula applies

$$\begin{aligned} v^2 &= {v_0}^2 - 2|a|\,(x - x_0) \\ 0 &= {v_0}^2 - 2\mu\, g\, S \\ {v_0}^2 &= 2\mu\, g\, S \\ S &= \frac{{v_0}^2}{2\mu g} \end{aligned}$$

3.6.5 Sliding on a Moving Platform

The problem is more intricate in case the platform can move horizontally in an ideal surface (see figure (3.6)). The platform has M mass. Now we have to take into account the reaction force of the friction force as well. This reaction force is accelerating the movable platform. How much distance ($S^\star$) does the block take on the platform?

During the sliding process the block is slowing down with a constant acceleration (μg), while the movable platform is accelerating with a different value ($\mu g m/M$). At that time instant (τ) when the speed of both objects is the same the friction force vanishes. Then both objects move at the same u speed

$$u = v_0 \frac{m}{m + M}$$

This result is completely analogous to the conservation of linear momentum (Laws of Conservation). Without detailed calculations the displacement of the block on the

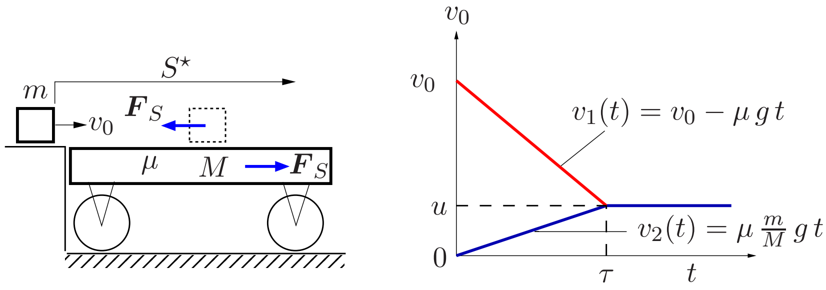


Figure 3.6: Slowing down on a movable platform.

movable platform is

$$S^\star = \frac{{v_0}^2}{2\mu\, g}\,\frac{M}{m+M} \tag{3.29}$$

If the mass of the platform is much greater compared to the mass of the block the platform is practically immobile. In this case $S^\star$ must be as great as S. Let us check it out.

$$S^\star = S\,\frac{M}{m+M}$$
$$\frac{S^\star}{S} = \frac{M}{m+M} = \frac{1}{1+\frac{m}{M}}$$
$$\lim_{M\to\infty}\left(\frac{S^\star}{S}\right) = \lim_{M\to\infty}\left(\frac{1}{1+\frac{m}{M}}\right) = 1\checkmark$$

3.7 Work-Energy Theorem

For the sake of simplicity we will study the motion of a particle in 1D. We assume that the force acting on the particle is constant. For this reason one of our kinematic formula applies: $v^2 = {v_0}^2 + 2a(x - x_0)$. Instead of x_0 and x we have introduced x_1 and x_2 notations.

$${v_2}^2 = {v_1}^2 + 2a(x_2 - x_1) \tag{3.30}$$

$${v_2}^2 - {v_1}^2 = 2\,\underbrace{\frac{F}{m}}_{a}\,\Delta\,x \qquad \left(\times\frac{m}{2}\right) \tag{3.31}$$

$$\underbrace{\frac{m{v_2}^2}{2}}_{E_2} - \underbrace{\frac{m{v_1}^2}{2}}_{E_1} = \underbrace{F\cdot\Delta\,x}_{W} \tag{3.32}$$

$$E_k = \frac{m\,v^2}{2} \tag{3.33}$$

This quantity is called the *kinetic energy* of the particle. Multiplying the numerator and the denominator by m and taking into consideration that the product mv equals

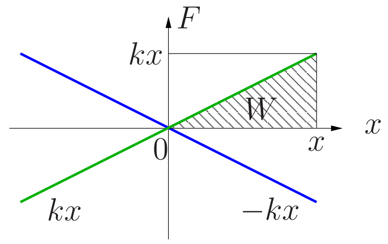


Figure 3.7: Force versus displacement of an ideal spring.

the momentum p of a body, the expression for the kinetic energy can be given the form

$$E_k = \frac{p^2}{2m} \tag{3.34}$$

It is clear from the above that work characterizes the change in energy due to the action of a force on a moving particle. The right hand side of the equation 3.32 is called work. *Although, work is energy transfer there is a fundamental difference between work and energy. Energy characterizes the instant state of a system, the work is an energy transfer between two states of the system.* The $F\Delta x$ form only applies when the magnitude of the force is constant and the direction of the displacement vector is parallel with F. This latter requirement is always true for one-dimensional motion, however, the magnitude of F can be altered during the motion. Let us calculate the work done during deformation of an ideal spring. In $F-x$ plane the work is numerically equal to the area under the $F(x)$ function. In accordance with figure (3.8) the area under the graph (in this special case) forms a rectangular triangle. Hence, the required work for deforming an ideal spring by an arbitrary value of x is

$$W = \frac{1}{2}kx \cdot x = \frac{1}{2}kx^2$$

When $F(x)$ is a general function ($F(x) = b \cdot x^2$) the area (Work) under the graph can be calculated by the following integral:

$$W_{12} = \int_1^2 F(x)\,dx$$

Equation 3.32 can be rewritten as:

$$K_2 - K_1 = W_{12} \tag{3.35}$$

where we introduced a new notation K for kinetic energy. Equation 3.35 is called *work-energy theorem* and in certain cases it is an extremely powerful tool for solving problems. It can be proven that the Equation 3.35 can be written in differential form as follows:

$$d\left(\frac{mv^2}{2}\right) = F(x)\,dx \tag{3.36}$$

We assume that $F(x)$ is a smooth and integrable function. Let us integrate it from position 1 to positon 2:

$$\int_1^2 d\left(\frac{mv^2}{2}\right) = \int_1^2 F(x)\,dx \tag{3.37}$$

The left-hand side is the difference between the values of the kinetic energy at position 2 and 1

$$\frac{m{v_2}^2}{2} - \frac{m{v_1}^2}{2} = G(2) - G(1), \tag{3.38}$$

where G is a *primitive function* (antiderivative) of the integrable function $F(x)$ over the interval $[1, 2]$ hence $F(x) = dG/dx$.

$$K_2 - K_1 = G_2 - G_1 \tag{3.39}$$

Introducing V the potential energy as $V = -G$

$$K_2 - K_1 = -V_2 - (-V_1) \tag{3.40}$$

$$K_1 + V_1 = K_2 + V_2 \tag{3.41}$$

The equation 3.41 is referred to as the conservation of mechanical energy. One of the most important notions in the framework of classical mechanics.

In this section the relationship between work and potential energy is presented in more detail. Recall the definition of the potential energy

$$F(x) = -\frac{dV(x)}{dx}$$

The potential energy in the Earth's gravitational field:

$$V(y) = mgy \quad /\tfrac{dV(y)}{dy} = -mg/$$

The potential energy in the elastic field of a spring:

$$V(x) = \frac{1}{2}kx^2 \quad /-\tfrac{dV(y)}{dy} = -kx/$$

3.7.1 Work and Potential Energy in Higher Dimension

If a particle moves in a plane the force may depend on $x-$ and $y-$ coordinates as well. If the force $\boldsymbol{F}(x, y)$ acts on a particle on a plane, its kinetic energy does not remains constant. The quantity

$$dW = \boldsymbol{F} \cdot d\boldsymbol{s} \tag{3.42}$$

is called the *work* done by the force $\boldsymbol{F}$ over the path $d\boldsymbol{s}$. The scalar product can be represented as the product of the projection of the force onto the direction of the displacement F_s and the elementary displacement $d\boldsymbol{s}$. Consequently, we can write that

$$dW = F_s\,ds = |\boldsymbol{F}||d\boldsymbol{s}|\cos(\theta) \tag{3.43}$$

If we know the components of the force (F_x, F_y) and the displacements (dx, dy) the result of the scalar products reads as:

$$dW = F_x \cdot dx + F_y \cdot dy \tag{3.44}$$

In higher dimensions it consists of energies of interaction of the particles in pairs. The definition of potential energy in higher dimension is

$$F(x, y, z) = -\nabla V(x, y, z),$$

where the so-called "nabla operator" denotes the vector differential operator. The notation grad$V$ is also commonly used for the gradient.

In the three-dimensional Cartesian coordinate system, the differential operator is given by

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

The gradient of the potential energy reads as:

$$\nabla V = \boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the standard unit vectors in the directions of the x, y and z coordinates, respectively.

Let us consider, in particular, the case of a particle of mass m moving under the gravitational influence of a point mass M. The position M is held fixed. Then

$$\boldsymbol{F} = -G\frac{Mm}{r^2}\boldsymbol{e}_r$$

where r is the distance between M and m, and $\boldsymbol{e}_r$ is a unit vector pointing from M to m.

The Work-Energy Theorem, asserts

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r}$$

where the right hand side is the total work done on the particle as it goes from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$. The path travelled by m is some curve which can be broken up into many small steps being represented by the vector $\Delta\boldsymbol{r}$. We can decompose $\Delta\boldsymbol{r}$ into two pieces, one parallel to $\boldsymbol{e}_r$ and the other perpendicular to $\boldsymbol{e}_r$. When we calculate the work $\boldsymbol{F} \cdot \Delta\boldsymbol{r}$, only the piece of $\Delta\boldsymbol{r}$ parallel to $\boldsymbol{e}_r$ contributes to the dot product. If we introduce polar coordinates, then the vector $\Delta\boldsymbol{r}$ goes from the point whose polar coordinates are r, θ, ϕ to the nearby point $(r + \Delta r, \theta + \Delta\theta, \phi + \Delta\phi)$. The piece of $\Delta\boldsymbol{r}$ along the radial direction is $(\Delta r)\boldsymbol{e}_r$, and thus we find

$$\boldsymbol{F} \cdot \Delta\boldsymbol{r} = \left(-G\frac{Mm}{r^2}\boldsymbol{e}_r\right) \cdot (\Delta r)\,\boldsymbol{e}_r = -G\frac{Mm}{r^2}\Delta r$$

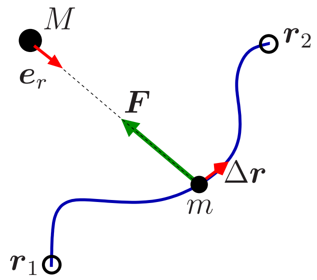


Figure 3.8: Work done by gravity.

since $\boldsymbol{e}_r \cdot \boldsymbol{e}_r = 1$.

Letting $\Delta r \to 0$ and adding up the contributions from all the steps, we find

$$\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot d\boldsymbol{r} = -\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} G\frac{Mm}{r^2} dr = G\frac{Mm}{r_2} - G\frac{Mm}{r_1}$$

This calculation shows that the work done by gravity depends only on the end points of the path, and is therefore the same for all paths between those end points. In accordance with the Work-Energy Theory we find

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = G\frac{Mm}{r_2} - G\frac{Mm}{r_1}$$

or equivalently

$$\frac{1}{2}mv^2 - G\frac{Mm}{r} = \text{constant}.$$

In this case we call $-GMm/r$ the gravitational potential energy since the sum of this quantity and the kinetic energy remain constant.

The property of the gravitational force which enabled us to define a potential energy is the following: the work done by gravity on a particle travelling between two points does not depend on the path.

3.7.2 Principle of Conservation of Mechanical Energy

The conservation of mechanical energy theorem is true if and only if

$$\oint \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r} = 0$$

When friction forces are present in a system, the total mechanical energy of the system diminishes (dissipates), transforming into non-mechanical forms of energy (heat). Forces leading to the dissipation of energy are called *dissipative*. According to Equation 3.41 *the total mechanical energy of a system of bodies on which only conservative forces act remains constant.* If non-conservative forces, for example, forces of friction, act in a closed system in addition to conservative ones the total mechanical energy is not conserved, however, the work-energy theorem can be used in such cases.

Consequently, if all the forces which do work are conservative, the work-energy theorem yields

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = V(\boldsymbol{r}_1) - V(\boldsymbol{r}_2)$$

which implies

$$\frac{1}{2}mv^2 + V(\boldsymbol{r}) = \text{constant} \tag{3.45}$$

The function $V(\boldsymbol{r})$ is called the potential energy and eqn. (3.45) is the statement of the *Principle of Conservation of Mechanical Energy.*

If a particle moves under the influence of a conservative force with associated potential V and also under the influence of a non-conservative force, then the work-energy theorem yields

$$K_1 + V_1 = K_2 + V_2 + W' \tag{3.46}$$

where W' is the work done by the non-conservative forces as the particle moves from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$. if the non-conservative force is a frictional drag directed opposite to the motion, then $W' > 0$.

We can pick some fixed point $\mathcal{R}$ called reference point. The reference point $\mathcal{R}$ is arbitrary. A change in the reference point causes potential energy at all points to change by an additive constant in V will not change anything.

3.7.3 Power

The work done in unit time is called *power*. If the work dW is done in the time dt, then the power is

$$P = \frac{dW}{dt} = \frac{d}{dt}(\boldsymbol{F}\cdot\boldsymbol{s}) = \dot{\boldsymbol{F}}\cdot\boldsymbol{s} + \boldsymbol{F}\cdot\dot{\boldsymbol{s}}$$

If $\boldsymbol{F}$ is a constant force

$$P = \boldsymbol{F}\cdot\boldsymbol{v}$$

There are machines, which have long been known, which can multiply the power of the human muscles. For example, the traditional bows and arbalests do the same work as the human body can do but much faster. These machines produce instantaneous power.

The instantaneous power of certain laser systems can reach as high as 1PW$=10^{15}$ J/s peak power. The steam turbines can produce 1.5 GW$= 1.5\cdot 10^9$ J/s continuous power.

3.7.4 Applications of the Work-Energy Theorem

As it can be seen in figure (3.9) we consider a system which consists of an ideal (massless) spring and a linked body. At the initial time moment the spring is undistorted, and the velocity of the joined body is zero. At a given moment a constant force is applied to the body. The friction force is ignored. How far does the body go compared to the initial position?

The typical misleading argument is the following: "Since the net force is a linearly decreasing function

$$F(x) = F - k\,x$$

in that position when the net force is zero the object is stopped." However, this conclusion is erroneous. *For this reason it is very important to emphasize that the net force determines the net acceleration and not the velocity.*

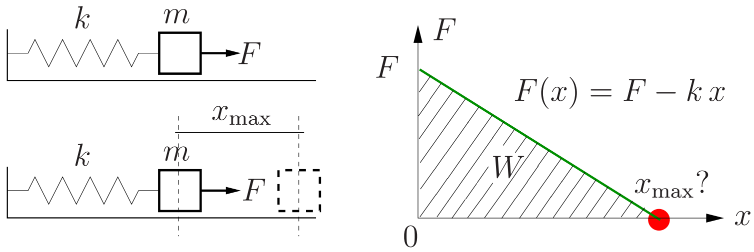


Figure 3.9: On the left: the spring-body system. On the right: the net force as a function of x.

Since both the initial and the final speeds are zero, the changing of the kinetic energy must also be zero. According to the work-energy theorem the total work must be equal to the kinetic energy changing.

$$\underbrace{\frac{m{v_2}^2}{2}}_{0} - \underbrace{\frac{m{v_1}^2}{2}}_{0} = \underbrace{W}_{0}$$

The net force during the interval $[0, F/k]$ is parallel with the displacement, hence the work has also positive sign. However in the interval $[F/k, x_{\max}]$ the net force and the displacement are oppositely directed. For this reason the sign of the work must be negative. The amount of positive work can be easily calculated

$$W = \frac{1}{2} F \cdot \frac{F}{k} = \frac{F^2}{2k}$$

Now it is obvious that the maximum displacement cannot be F/k because it would lead to nonzero net work in contradiction with the work-energy theorem. Consequently, the greatest distance must be twice as large as F/k, because any other choice of the maximum position would lead to nonzero total work and would violate the work-energy theorem as it can be seen in figure (3.10).

In the position $x = F/k$ the body has the highest speed. According to the work-energy theorem it can be calculated as follows:

$$\frac{F^2}{\cancel{2}k} = \frac{1}{\cancel{2}} m v^2$$

$$v_{\max} = \pm \frac{F}{\sqrt{km}}$$

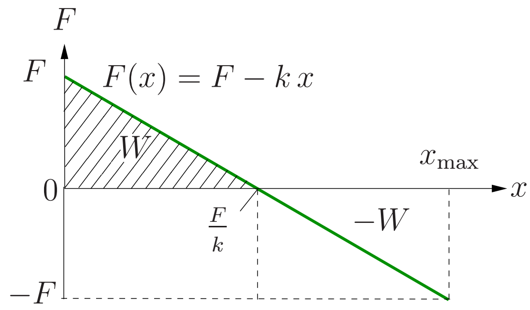


Figure 3.10: The net force as a function of displacement.

The maximum acceleration is connected to the maximum net force. The net force is maximum in $x = 0$ and $x = 2F/k$ positions. The values of the acceleration in these positions are

$$a_{\max} = \pm\frac{F}{m} \tag{3.47}$$

The positive sign indicates that the direction of the acceleration is parallel with the $x-$axis.

3.7.5 Applications of Conservation of Mechanical Energy

Le us consider again the former problem in the framework of conservation of mechanical energy. At the initial position the object has only potential energy in the "force field" of pulling force. This potential energy is similar to mgh. The conservation of mechanical energy in the initial and the farthest position of the body reads as:

$$\underbrace{K_0}_{0} + \underbrace{V_0^{\text{spring}}}_{0} + V_0^{\text{pull}} = \underbrace{K_{(x_{\max})}}_{0} + V_{(x_{\max})}^{\text{spring}} + \underbrace{V_{(x_{\max})}^{\text{pull}}}_{0}$$

$$\begin{aligned} V_0^{\text{pull}} &= V_{(x_{\max})}^{\text{spring}} \\ F\,\cancel{x_{\max}} &= \frac{1}{2}k\,x_{\max}^{\cancel{2}} \\ x_{\max} &= \frac{2F}{k} \end{aligned}$$

Hence the potential energy with respect to the position $x = 2F/k$ is $V = F\,(2F/k - x)$. At the initial position ($x = 0$) the potential energy has to agree with the total energy of the system for this reason

$$V(0) = F\,\frac{2F}{k} - 0 = \frac{2\,F^2}{k} \tag{3.48}$$

So, the potential energy function takes the following form:

$$V(x) = \frac{2\,F^2}{k} - F\,x$$

The energy equation of the system in an arbitrary x position takes the following form:

$$\frac{2\,F^2}{k} = \frac{1}{2}mv^2 + \frac{1}{2}k\,x^2 + \frac{2\,F^2}{k} - F\,x \tag{3.49}$$

If we eliminate the total energy from the equation we get the kinetic energy as a function of x.

$$\begin{aligned} \cancel{\frac{2\,F^2}{k}} &= \frac{1}{2}mv^2 + \frac{1}{2}k\,x^2 + \cancel{\frac{2\,F^2}{k}} - F\,x \\ \frac{1}{2}mv^2 &= F\,x - \frac{1}{2}k\,x^2 \end{aligned}$$

Mathematically the kinetic energy function is a parabola. The problem is more intricate if the spring is initially compressed (or stretched). Without detailed calculations we give some expression and functions. The energies in both cases are depicted in figure (3.11).

Type of Energy	Initially Undeformed Spring	Initially compressed Spring
E_{total}	$\frac{2\,F^2}{k}$	$\frac{2\,F^2}{k} + 2F\,x_0 + \frac{1}{2}kx_0{}^2$
$V(x)_{\text{spring}}$	$\frac{1}{2}k\,x^2$	$\frac{1}{2}k\,(x - x_0)^2$
$V(x)_{\text{pulling}}$	$\frac{2\,F^2}{k} - F\,x$	$\frac{2\,F^2}{k} + 2F\,x_0 - F\,x$
$K(x)$	$F\,x - \frac{1}{2}k\,x^2$	$k\,x_0\,x + F\,x - \frac{1}{2}k\,x^2$

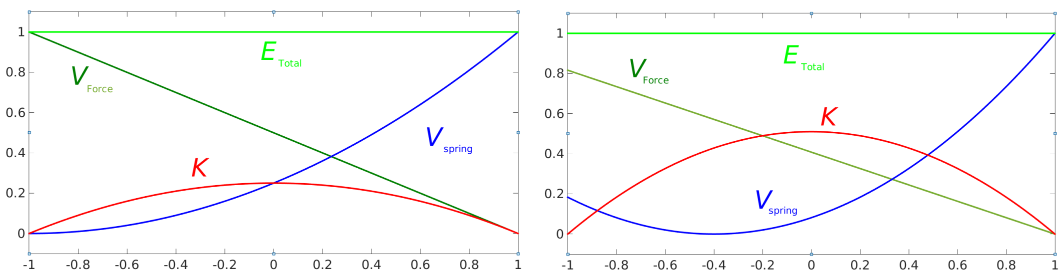


Figure 3.11: On the left: the energy balance of an initially undeformed spring-body system. On the right: the energy balance of an initially compressed spring-body system.

Let us test the energy equation by substituting $x = F/k$. Now we know that the body speed is maximal in that position.

$$\begin{aligned}
\frac{2\,F^2}{k} &= \frac{1}{2}mv^2 + \frac{1}{2}k\,x^2 + \frac{2\,F^2}{k} - F\,x \\
\cancel{\frac{2\,F^2}{k}} &= \frac{1}{2}mv^2 + \frac{1}{2}k\,\frac{F^2}{k^2} + \cancel{\frac{2\,F^2}{k}} - F\,\frac{F}{k} \\
0 &= \frac{1}{2}mv^2 + \frac{1}{2}\cancel{k}\,\frac{F^2}{k^{\cancel{2}}} + -F\,\frac{F}{k} \\
0 &= \frac{1}{\cancel{2}}mv^2 - \frac{F^2}{\cancel{2}k} \\
v &= \frac{F}{\sqrt{km}}
\end{aligned}$$

3.8 Laws of Conservation

Bodies forming a mechanical system may interact with one another and with bodies not belonging to the given system. Accordingly, the forces acting on the bodies of a system can be divided into *internal* and *external* ones. We shall define internal forces as the forces with which a given body is acted upon by the other bodies of the system, and external forces as those produced by the action of bodies not belonging to the system. If external forces are absent, the relevant system is called *closed*.

There are functions of the coordinates and velocities of the particles forming a system for closed systems that retain constant values upon motion. These functions are called *motion integrals*. There are three additive motion integrals. The first is called *energy*, the second—*momentum*, and the third—*angular momentum*.

Thus, three physical quantities do not change in closed systems, namely, energy, momentum, and angular momentum. Accordingly, there are three *laws of conservation*—that of energy conservation, that of momentum conservation, and that of angular momentum conservation. These laws are closely associated with the fundamental properties of space and time.

The conservation of energy is based on the *uniformity of time*, i. e. the equivalence of all moments of time. The equivalence should be understood in the sense that the substitution of the moment of time t_2 for the moment t_1 without a change in the values of the coordinates and velocities of the particles does not change the mechanical properties of a system. This signifies that after such a substitution, the coordinates and velocities of the particles have the same values at any moment of time t'_2 as they have had before the substitution at the moment t'_1.

The conservation of momentum is based on the *uniformity of space*, i. e. the identical properties of space at all points. This should be understood in the sense that translation of a closed system from one place in space to another without changing mutual arrangement and velocities of the particles does not change the mechanical properties of the system.

Finally, the conservation of angular momentum is based on the *isotropy of space*, i. e. the identical properties of space in all directions.

3.9 Collision of Two Bodies

When bodies collide with one another, they become deformed. The kinetic energy which the bodies had before the collision partially or completely transforms into the potential energy of the bodies. An increase in the internal energy of bodies is attended by elevation of their temperature.

Two extreme kinds of collisions are distinguished: perfectly elastic and completely inelastic ones. A *perfectly elastic collision* is one in which the mechanical energy of the bodies does not transform into other non-mechanical kinds of energy. In such a collision the kinetic energy transforms completely or partly into the potential energy of elastic deformation. Next the bodies return to their original shape, repelling each other. As a result, the potential energy of elastic deformation again transforms into kinetic energy, and the bodies fly apart. The magnitude and direction of the velocities are determined by two conditions—conservation of the total energy and conservation of the total momentum of the system of bodies.

3.9.1 Completely Inelastic Collision

A *completely elastic collision* is characterised by the fact that no potential energy of deformation is produced. The kinetic energy of the bodies completely or partly transforms into internal energy. After colliding, the bodies either move with the same velocity or are at rest. In a completely inelastic collision, only the law of conservation of momentum is observed. The law of conservation of mechanical energy is violated because of the presence of non-conservative forces.

Let us first consider a completely inelastic collision of two particles forming a closed system. Let the masses of the particles be m_1 and m_2, and their velocities before colliding $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. In view of the law of momentum conservation, the total momentum of the particles after the collision must be the same as before it

$$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = (m_1 + m_2)\,\boldsymbol{v} \tag{3.50}$$

$$\boldsymbol{v} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2} \tag{3.51}$$

$\boldsymbol{v}$ is the identical velocity of both particles after colliding. We can calculate the value of mechanical energy loss as follows:

$$\Delta E = \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_1v_1^2 - \frac{1}{2}(m_1 + m_2)\,v^2 = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}(v_1 - v_2)^2 \tag{3.52}$$

A ballistic pendulum is a device for measuring a bullet's momentum, from which it is possible to calculate the velocity and kinetic energy. A large wooden block suspended by two cords serves as the pendulum bob. When a bullet is fired into the bob, its momentum is transferred to the bob. The bullet's momentum can be determined from the amplitude of the pendulum swing. The velocity of the bullet, in turn, can be derived from its calculated momentum. Ballistic pendulums allow direct measurement of the projectile velocity. Because of its simplicity and usefulness it can be found as a popular demonstration tool. Unlike other methods of measuring the speed of a bullet, the basic calculations for a ballistic pendulum do not require any measurement of time, but rely only on measures of mass and distance.

After the collision, conservation of energy can be used in the swing of the combined masses upward, since the gravitational potential energy is conservative. Measuring the height of the swing revealed the speed of the bullet, but since the block was increasing in mass with the added bullets, the mass of the block had to be checked as well as the mass of the bullet being fired.

$$
\begin{aligned}
m\,v &= (m+M)\;U \\
U &= \frac{m}{m+M}\,v \\
\frac{1}{2}\,(m+M)\;U^2 &= (m+M)\,gh \\
v &= \left(\frac{m+M}{m}\right)\,\sqrt{2gh}
\end{aligned}
$$

Tool	Maximum speed	projectile mass
Bow/arbalest	125 m/s = 450 km/h	30-100 g
(AK 47-es) assault rifle	715 m/s = 2574 km/h	8 g
Dragunov sniper rifle	810 m/s = 2916 km/h	11.98 g
Gepárd M1 sniper rifle	840 m/s = 3024 km/h	45 g
D-20 towed gun-howitzer	650 m/s	20 kg
railgun	5900 m/s = 21 240 km/h	1 kg?

3.9.2 Elastic Collision

Let us now consider a completely elastic collision. We shall limit ourselves to the case of a central and head on collision of two homogeneous bodies. A collision is called *head on* if the bodies before colliding travelled along a straight line passing through their centres. A head on collision of two spheres can take place if the spheres are moving toward each other, or if one of the spheres is overtaking the other one. We shall assume that the spheres form a closed system or that the external forces applied to them balance each other. We shall also assume that the spheres do not rotate.

$$m_1\,\boldsymbol{v}_1 + m_2\,\boldsymbol{v}_2 = m_1\,\boldsymbol{u}_1 + m_2\,\boldsymbol{u}_2 \tag{3.53}$$

$$\frac{1}{2}m_1\,{\boldsymbol{v}_1}^2 + \frac{1}{2}m_2\,{\boldsymbol{v}_2}^2 = \frac{1}{2}m_1\,{\boldsymbol{u}_1}^2 + \frac{1}{2}m_2\,{\boldsymbol{u}_2}^2 \tag{3.54}$$

$$
\begin{aligned}
\boldsymbol{u}_1 &= \frac{(m_1-m_2)\,\boldsymbol{v_1} + 2m_2\boldsymbol{v}_2}{m_1+m_2} \\
\boldsymbol{u}_2 &= \frac{(m_2-m_1)\,\boldsymbol{v}_2 + 2m_1\boldsymbol{v}_1}{m_1+m_2}
\end{aligned}
$$

We can use the former equations to find the velocity of a sphere after an elastic collision with a stationary or a moving wall. We assume that the wall is stationary $\boldsymbol{v}_2 = 0$.

$$\boldsymbol{u}_1 = \frac{2m_2}{m_1+m_2}\boldsymbol{v}_2 + \frac{m_1-m_2}{m_1+m_2}\boldsymbol{v}_1$$
$$\lim_{m_2\to\infty}\boldsymbol{u}_1 = \lim_{m_2\to\infty}\frac{2}{\frac{m_1}{m_2}+1}\boldsymbol{v}_2 + \frac{\frac{m_1}{m_2}-1}{\frac{m_1}{m_2}+1}\boldsymbol{v}_1 = 2\underbrace{\boldsymbol{v}_2}_{0} - \boldsymbol{v}_1 = -\boldsymbol{v}_1$$

Now we assume that the wall is moving oppositely directed as the particle moves $-\boldsymbol{v_2}$.

$$\boldsymbol{u}_1 = -\frac{2m_2}{m_1+m_2}\boldsymbol{v}_2 + \frac{m_1-m_2}{m_1+m_2}\boldsymbol{v}_1$$
$$\lim_{m_2\to\infty}\boldsymbol{u}_1 = \lim_{m_2\to\infty}\left(-\frac{2}{\frac{m_1}{m_2}+1}\boldsymbol{v}_2 + \frac{\frac{m_1}{m_2}-1}{\frac{m_1}{m_2}+1}\boldsymbol{v}_1\right) = -2\boldsymbol{v}_2 - \boldsymbol{v}_1$$

The result obtained shows that the velocity of the wall remains unchanged. The velocity of the particle, however, if the wall is stationary, reverses. If the wall is moving, the magnitude of the velocity of the particle also changes. It grows by $2v_2$ if the wall moves toward the sphere.

Let us consider the case when the two masses of the colliding particles are equal: $m_1 = m_2$. *The particles exchange velocities when they collide.*

3.10 Planetary Motion

At the beginning of the seventeenth century, Johannes Kepler (1571-1630) proposed three laws to describe the motion of the planets. These laws predated Newton's laws of motion and his law of gravity. They offered a far simpler description of planetary motion than anything that had been proposed previously.

Kepler's laws of planetary motion are

- The planets travel in elliptical orbits with the Sun at one focus of the ellipse.
- A line drawn from a planet to the Sun sweeps out equal areas in equal time intervals.
- The square of the orbital period is proportional to the cube of the average distance from the planet to the Sun.

Kepler's first law can be derived from the inverse square law of gravitational attraction. A derivation is a bit complicated. We can derive Kepler's third law from Newton's law of universal gravitation for the special case of a circular orbit.

$$\sum \boldsymbol{F} = m\boldsymbol{a}$$
$$\frac{GmM}{r^2} = \frac{mv^2}{r}$$

Solving for v yields

$$v = \sqrt{\frac{GM}{r}}$$

The distance travelled during one revolution is the circumference of the circle, which is equal to $2\pi r$. The speed is the distance travelled during one orbit divided by the period:

$$\sqrt{\frac{GM}{r}} = \frac{2\pi r}{T}$$

Now we solve for T:

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

Squaring both sides yields:

$$T^2 = \frac{4\pi r^3}{GM} = \text{constant} \cdot r^3 \tag{3.55}$$

Equation (3.55) is Kepler's third law: the square of the period of a planet is directly proportional to the cube of the radius. In case of elliptical motion r means average orbital radius.

Planetary orbits are affected by gravitational interactions with other planets: Kepler's law ignores these small effects. Although Kepler's laws were derived for the motion of planets, they apply to satellites orbiting the Earth as well.

3.11 Determinism

French mathematician Pierre Laplace speculated that Newtonian mechanics heralded a rigid determinism that would theoretically enable the successful prediction of the entire future of the universe, given absolute knowledge of its complete state at any given time. Laplace expressed this as follows: "We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

Chaos theory concerns deterministic systems whose behaviour can in principle be predicted. Chaotic systems are predictable for a while and then appear to become random. Small differences in initial conditions (such as those due to rounding errors in numerical computation) yield widely diverging outcomes for such dynamical systems, rendering long-term prediction of their behaviour impossible in general. This happens even though these systems are deterministic, meaning that their future behaviour is fully determined by their initial conditions, with no random elements involved. In other words, the deterministic nature of these systems does not make them predictable.

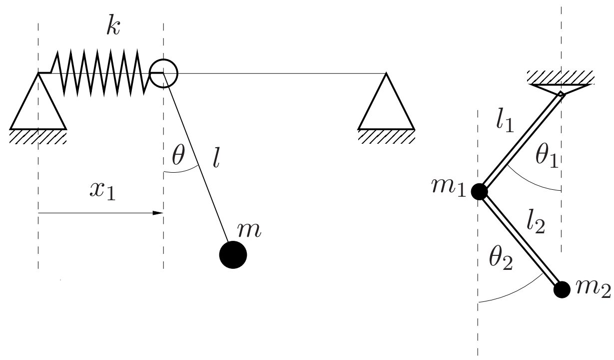


Figure 3.12: A simple pendulum driven at the suspension point (on the left) and a double pendulum (on the right).

This behaviour is known as deterministic chaos, or simply chaos. The theory was summarized by Edward Lorenz as: Chaos: When the present determines the future, but the approximate present does not approximately determine the future. A consequence of sensitivity to initial conditions is that if we start with a limited amount of information about the system (as is usually the case in practice), then beyond a certain time the system is no longer predictable.

A simple pendulum driven harmonically at the suspension point. Despite its simplicity, it generally does extremely complicated motion. The general behaviour of the motion is chaotic. The equations of motion are the following:

$$\ddot{x}_1 \cos\theta - \dot{x}_1 (\sin\theta)\, \dot{\theta} + l\ddot{\theta} + g\sin\theta = 0 \tag{3.56}$$

$$\ddot{x}_1 + l\ddot{\theta}\cos\theta - l\dot{\theta}^2 \sin\theta + \frac{k}{m}x_1 = 0 \tag{3.57}$$

In the last chapter we will show the method (Lagrangian mechanics) by witch these equations can be obtained.

The double pendulum also undergoes chaotic motion, and shows a sensitive dependence on initial conditions. The equations of motions are the following:

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + l_2 m_2 g \sin\theta_2 = 0 \tag{3.58}$$

$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \tag{3.59}$$

In the framework of Lagrangian mechanics these equations can be deduced, however the solution in both cases required sophisticated numerical methods that can be performed on a computer.

Chapter 4

Mechanics of a Rigid Body

4.1 Rotation

Let us consider a point particle rotating around a fixed axis (see figure 4.1). Introducing the rotational energy as follows:

$$K_{\mathrm{rot}} = \frac{1}{2}\, I\, \omega^2$$

where I is the moment of inertia of the body about the axis $\hat{\boldsymbol{\jmath}}$. I is sometimes called the "rotational inertia" of the body. This is excellent terminology since I is indeed the measure of how hard it is to change the angular velocity of a body. However in this particular case the kinetic energy K and the rotational energy K_{rot} are the same. Hence

$$K = \frac{1}{2} m\, v^2 = \frac{1}{2} m\, (R\omega)^2$$
$$K_{\mathrm{rot}} = \frac{1}{2} I\, \omega^2$$

Equating the right hand sides of the equations we get the moment of inertia for a

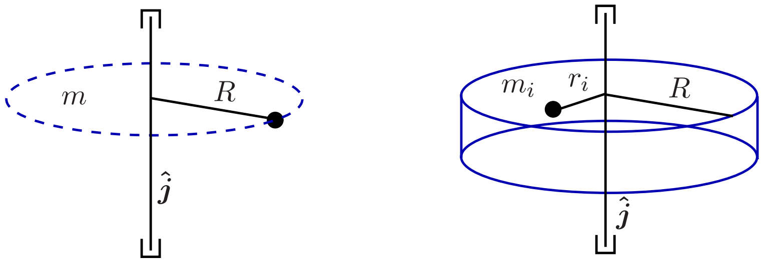


Figure 4.1: Rotation around fixed axis. On the left: a point particle. On the right: a rigid cylinder.

point particle:

$$I = m\, R^2$$

When a rigid object is rotating in place, it has kinetic energy because each particle other than those on the axis of rotation is moving in a circle around the axis. We can calculate the kinetic energy of rotation by summing the kinetic energy of each particle.

If a rigid object consists of N particles, the sum of the kinetic energies of the particles can be written mathematically using a subscript to label the mass and speed of each particle:

$$K_{\text{rot}} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots + \frac{1}{2}m_Nv_N^2 = \sum_{i=1}^{N}\frac{1}{2}m_iv_i^2$$

Let us begin the consideration of the rotation of a symmetric body about fixed axis, which we shall call the z-axis. The linear velocity of the elementary mass m_i is $v_i = \omega r_i$, where r_i is the distance from the mass m_i to the z-axis. Consequently, we get the following expression for the kinetic energy of the i-th elementary mass:

$$K_i = \frac{1}{2}m_i\omega^2\, r_i^2$$

The entire object rotates at the same angular velocity ω, so the constants $1/2$ and ω^2 can be factored out of each term of the sum:

$$K_{\text{rot}} = \frac{1}{2}\,\omega^2\left(\sum_{i=1}^{N} m_i\, r_i^2\right)$$

The quantity of the parentheses cannot change since the distance between each particle and the rotation axis stays the same if the object is rigid and doesn't change shape. However difficult it may be to compute this summation. The moment of inertia can be written in the form:

$$I = \sum_{i=1}^{N} m_i\, r_i^2$$

4.1.1 Moment of Inertia

As we have mentioned I is the measure of how hard it is to change the angular velocity of a body just as m is the measure of how hard it is to change the linear velocity. The elementary mass Δm_i equals the product of the density of a body ρ_i at a given point and the corresponding elementary volume ΔV_i:

$$\Delta m_i = \rho_i \Delta m_i$$

Consequently, the moment of inertia can be written in the form:

$$I = \sum_{i=1}^{N} \rho_i\, r_i^2 \Delta V_i \tag{4.1}$$

If the density of a body is constant, it can be taken outside the sum:

$$I = \rho \sum_{i=1}^{N} r_i^2 \Delta V_i \tag{4.2}$$

Equations (4.1) and (4.2) are approximate. Their accuracy grows with diminishing elementary volumes ΔV_i and the elementary masses Δm_i corresponding to them. Hence, the task of finding the moments of inertia consists in integration:

$$I = \lim_{\Delta V_i \to 0} \sum_{i=1}^{N} \rho_i \, r_i^2 \Delta V_i = \int_M \rho \, r^2 \, dV \tag{4.3}$$

If the density of a body is constant, it can be taken outside of the integral:

$$I = \rho \int_M r^2 \, dV \tag{4.4}$$

The integral in Equation (4.4) is taken over the entire volume of the body. The quantities ρ and r in these integrals are position functions, i. e., for example, functions of the Cartesian coordinates x, y, and z. Let us find the moment of inertia of homogeneous disk relative to an axis perpendicular to the plane of the disk and passing through its centre. Let us divide the disk into annular layers of thickness dr (see figure

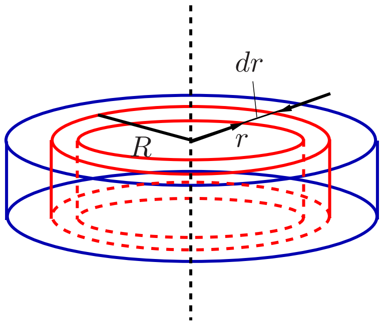


Figure 4.2: An infinitesimal layer of a disk.

(4.2)). All the points of one layer will be at the same distance r from the axis. The volume of such a layer is

$$dV = b 2\pi r dr$$

where b is the thickness of the disk. Since the disk is homogeneous, its density at all of its points is the same can be put outside the integral:

$$I = \rho \int_M r^2 \, dV = \rho \int_0^R r^2 b 2\pi r \, dr \tag{4.5}$$

where R is the radius of the disk. Let us put the constant factor $b2\pi$ outside the integral:

$$I = b 2\pi \rho \int_0^R r^3 \, dr = b 2\pi \rho \frac{R^4}{4} \tag{4.6}$$

Finally, introducing the mass of the disk m equal to the product of the density ρ and the volume of the disk $b\pi R^2$, we get

$$I = \frac{mR^2}{2} \tag{4.7}$$

The finding of the moment of inertia in the above example was simplified quite considerably owing to the fact that the body was homogeneous and symmetrical. If we wanted to find the moment of inertia of a disk relative an arbitrary axis the calculation would be much more complicated.

We can calculate the moment of inertia of a hoop with radius R and width δ as follows:

$$\begin{aligned}
I &= b2\pi\rho \int\limits_{R-\delta}^{R} r^3\,dr = 2\pi\rho b \frac{R^4}{4}\bigg|_{R-\delta}^{R} \\
I &= \frac{1}{2}\pi\rho b\left(R^4 - (R-\delta)^4\right) \\
I &= \frac{1}{2}\pi\rho b\left(R^4 - \left(\left(R^2 - 2R\delta + \cancel{\delta^2}\right)\left(R^2 - 2R\delta + \cancel{\delta^2}\right)\right)\right) \\
I &= \frac{1}{2}\pi\rho b\left(R^4 - \left(R^4 - 2R^3\delta - 2R^3\delta + \cancel{4R^2\delta^2}\right)\right) \\
I &= \frac{1}{2}\pi\rho b\left(\cancel{R^4} - \cancel{R^4} + 4R^3\delta\right) \\
I &= \frac{1}{2}\left(2\pi\rho b\,\delta\right)2R^2 = mR^2
\end{aligned}$$

We can calculate the moment of inertia of a thick ring with inner radius R_1 and outer radius R_2 as follows:

$$\begin{aligned}
I &= I_2 - I_1 = \frac{1}{2}m_2R_2^2 - \frac{1}{2}m_1R_1^2 \\
I &= \frac{1}{2}R_2^2\pi b\rho R_2^2 - \frac{1}{2}R_1^2\pi b\rho_2 R_1^2 \\
I &= \frac{\pi\rho b}{2}\left(R_2^4 - R_1^4\right) \quad /\rho = \frac{m}{V} = \frac{m}{\pi b\left(R_2^2 - R_1^2\right)}/ \\
I &= \frac{1}{2}\frac{m}{\cancel{R_2^2 - R_1^2}}\cancel{\left(R_2^2 - R_1^2\right)}\left(R_2^2 + R_1^2\right) \\
I &= \frac{1}{2}m\left(R_2^2 + R_1^2\right)
\end{aligned}$$

Finally we calculate the moment of inertia of a *uniform rod* (mass M, length L) about an end. A small portion of the rod, of length dx, has mass (M/Ldx) (see Fig.4.3). If x is measured from the end of the rod, we find

$$I = \frac{M}{L}\int\limits_{0}^{L} x^2 dx = \frac{1}{3}ML^2 \tag{4.8}$$

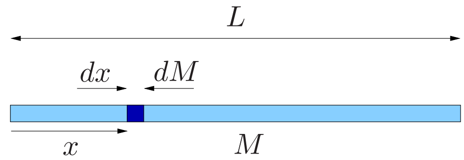


Figure 4.3: A rigid rod of length L and mass M.

Similarly, the moment of inertia of the rod about its midpoint is

$$I = \frac{M}{L}\int_{L/2}^{L/2} x^2 dx = \frac{1}{12}ML^2 \tag{4.9}$$

4.1.2 Parallel Axis Theorem

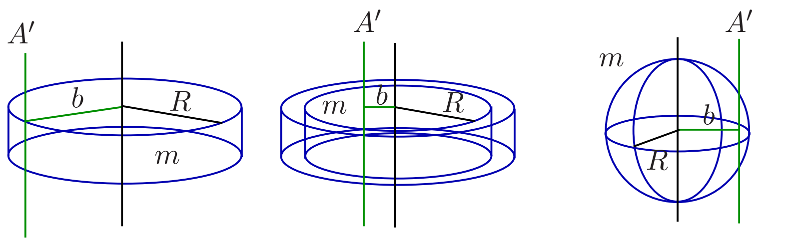
The finding of the moment of inertia is considerably simplified in such a cases if we uses the *Steiner* or *parallel axis theorem*, which formulated as follows: *the moment of inertia I relative to an arbitrary axis equals the moment of inertia I_c relative to an axis parallel to the given one and passing through the body's centre of mass plus the product of the body's mass m and the square of the distance b between the axis:*

$$I = I_c + mb^2$$

According to the parallel axis theorem (in accordance with figure (4.4)) , the moment of inertia of the disk relative to the axis A' equals the moment of inertia relative to the axis passing through the centre of the disk, which we have found

$$I^{\star} = \frac{mR^2}{2} + mR^2$$

As it can be seen in figure (4.5) many mechanical systems consist of rigid bodies which



$I^{\star} = \frac{1}{2}m\,R^2 + mR^2$ $\quad$ $I^{\star} = m\,R^2 + mb^2$ $\quad$ $I^{\star} = \frac{2}{5}m\,R^2 + mb^2$

Figure 4.4: Moments of inertia calculated by the parallel axis theorem.

can be revolved around a fixed axis. For example, doors, windows, and levers. We can simplify those objects as a rod.

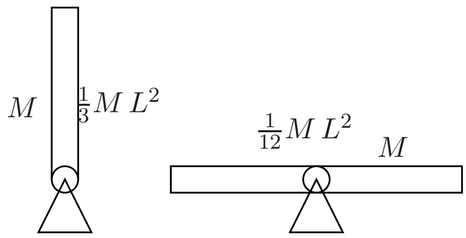


Figure 4.5: Moments of inertia of a rigid rod.

4.1.3 Torque

A quantity related to force, called *torque*, plays the role in rotation that force itself plays in translation. A torque is not separate form a force; it is impossible to exert a torque without exerting a force. Torque is a measure of how effective a given force is at twisting or turning something. For something rotating about a fixed axis a torque can *change* the rotational motion either by making it rotate faster or by slowing it down.

To satisfy the requirements of the daily life, we define the magnitude of the torque as the product of the distance between the rotation axis and the point of application of the force (r) with the perpendicular component of the force ($F_\perp$).

$$\tau = \pm r F_\perp$$

The symbol for torque is τ, the Greek letter tau. The SI unit of torque is the N·m. The SI unit of *energy*, the joule, is equivalent to N·m, but we do not write torque in joules. Even though both energy and torque can be written using the same SI base units, the two quantities have different meaning. Torque is not a form of energy. In vectorial form as the following:

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

There is another, completely equivalent, way to calculate torques that is often more

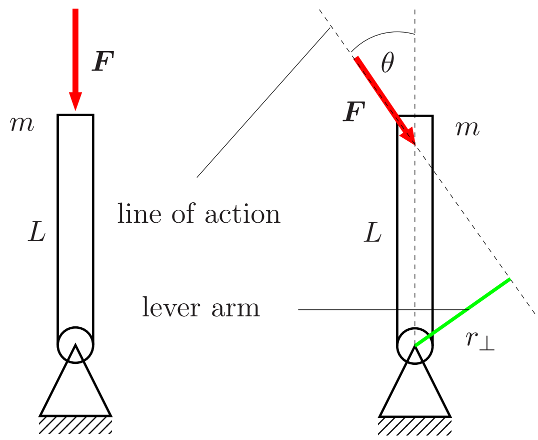


Figure 4.6: Line of action and lever arm.

convenient than finding the perpendicular component of the force. The distance $r_\perp$ is called the *lever arm* or *moment of arm*. The magnitude of the torque is, therefore, the magnitude of the force times lever arm.

$$\tau = \pm r_\perp F$$

In order to find the lever arm draw a line parallel to the force through the force's point of application. This line is called the force's *line of action*. As it can be seen in figure (4.6). The distance from the axis to the line of action along this perpendicular line is the lever arm ($r_\perp$). If the line of action of the force goes through the rotation axis, the lever arm and the torque are both zero.

The torque is a vector quantity hence a vector product can even better characterise its properties

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

Hence $\boldsymbol{\tau}$ a vector that is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{F}$ thus normal to the plane containing them. The magnitude of the cross product is

$$|\boldsymbol{\tau}| = rF \sin\theta$$

4.1.4 Work done by the torque

Torques can do work. Actually, it is the force that does the work, but in rotational problems it is often simpler to calculate the work done from the torque. Just as the work done by a constant force is the product of force and the parallel component of displacement, work done by a constant torque can also be calculated as the torque times the *angular* displacement.

Imagine a torque acting on a wheel that spins through an angular displacement $\Delta\varphi$ while the torque is applied. The work done by the force that gives rise to the torque is a product of the perpendicular component of the force ($F_\perp$) with the arc length s. We use the perpendicular force component because that is the component parallel to the displacement, which is instantaneously tangent to the arc of the circle. Thus,

$$W = F_\perp s = \frac{\tau}{r} \cdot r\Delta\varphi = \tau\Delta\varphi$$

In case the magnitude of torque is a function of φ

$$W = \int\limits_{\varphi_1}^{\varphi_2} \tau(\varphi)\, d\varphi$$

4.1.5 Rotation form of Newton's second law

All of the rotational concepts covered are analogous to linear motion properties, with very similar equations.

$$\int_{x_1}^{x_2} F(x)dx = \frac{1}{2}m\,{v_2}^2 - \frac{1}{2}m\,{v_1}^2 \;\rightleftarrows\; \underbrace{\int_{\varphi_1}^{\varphi_2} \tau(\varphi)d\varphi}_{W} = \underbrace{\frac{1}{2}I\,{\omega_2}^2}_{K_{\mathrm{rot}}2} - \underbrace{\frac{1}{2}I\,{\omega_1}^2}_{K_{\mathrm{rot}}1}$$

According to work-energy theorem for rotation, the amount of work done by all the torques acting on a rigid body under a fixed axis rotation (pure rotation) equals the change in its rotational kinetic energy:

$$\int_{\varphi_1}^{\varphi_2} \tau\,(\varphi)\,d\varphi = \frac{1}{2}I\,{\omega_2}^2 - \frac{1}{2}I\,{\omega_1}^2$$

If a constant torque acting on a rigid object and the moment of inertia do not change during the process the work-energy theorem takes a simpler form as follow:

$$\tau\,(\varphi_2 - \varphi_1) = \frac{1}{2}I\,\left({\omega_2}^2 - {\omega_1}^2\right) \tag{4.10}$$

We assume that the angular acceleration is constant hence one of our kinetic formula applies

$$ {\omega_2}^2 = {\omega_1}^2 + 2\alpha\,(\varphi_2 - \varphi_1) \tag{4.11}$$

$$ \left({\omega_2}^2 - {\omega_1}^2\right) = 2\alpha\,(\varphi_2 - \varphi_1) \tag{4.12}$$

If we substitute the equation 4.12 into equation 4.10 we get:

$$\tau\cancel{(\varphi_2 - \varphi_1)} = \frac{1}{\cancel{2}}I\,\cancel{2}\alpha\cancel{(\varphi_2 - \varphi_1)}$$
$$\tau = I\,\alpha$$

The angular acceleration of the body about its centre of mass will be according to the equation

$$\sum_i \tau_i = I\,\ddot{\varphi}$$

An object at rest tends to remain at rest and an object in rotation tends to continue rotating with constant angular velocity unless compelled by a net external torque to act otherwise.

Consider a particle of mass m whose position vector with respect to the origin O of an inertial reference frame is $\boldsymbol{r}$ and whose velocity and acceleration $\boldsymbol{v} = d\boldsymbol{r}/dt$ and $\boldsymbol{a} = d^2\boldsymbol{r}/dt^2$. Taking the cross product of both sides of the equation of motion $\boldsymbol{F} = m\boldsymbol{a}$ with $\boldsymbol{r}$ we obtain

$$\boldsymbol{r} \times \boldsymbol{F} = m\boldsymbol{r} \times \boldsymbol{a} \tag{4.13}$$

where $\boldsymbol{F}$ is the total force acting on the particle. The left side of eqn. (4.13) is the torque $\boldsymbol{\tau}$ (about the origin O) acting on the particle. We also define the *angular momentum* $\boldsymbol{L}$ of the particle about the origin O by the equation

$$\boldsymbol{L} = m\boldsymbol{r} \times \boldsymbol{v} \tag{4.14}$$

Using the rule for differentiating a cross-product we find:

$$\dot{\boldsymbol{L}} = m\boldsymbol{v} \times \boldsymbol{v} + m\boldsymbol{r} \times \boldsymbol{a}. \tag{4.15}$$

Since $\boldsymbol{v} \times \boldsymbol{v} = 0$ we get:

$$\tau = \dot{\boldsymbol{L}} \tag{4.16}$$

$$\sum_i \boldsymbol{F}_i = \dot{\boldsymbol{p}} \quad \leftrightarrows \quad \sum_i \boldsymbol{\tau}_i = \dot{\boldsymbol{L}}$$

4.1.6 *L* in central force field

Central forces are very important in physics and engineering. The central force is always directed from m toward, or away from a fixed point O. the magnitude of the force only depends on the distance r from O. Forces having these properties are called central forces.

A particle of mass m moves on the surface of a smooth horizontal table, constrained by a string which passes through a hole in the table (Fig.4.7). Initially the particle is moving with speed v_1 in a circle of radius r_1. The string is pulled slowly until the particle is moving in a smaller circle of radius r_2. If we apply the work-energy theorem

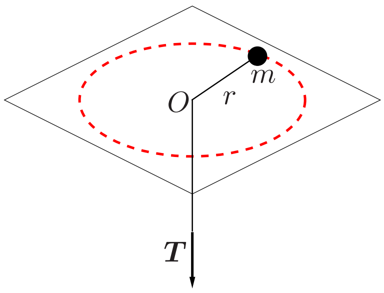


Figure 4.7: A particle moves on a horizontal table in a circular path.

to the infinitesimal process in which the length of the string is changed from r to $r+dr$ and the speed of the particle changes from $v+dv$ we find:

$$-Tdr = K_2 - K_1 \tag{4.17}$$

$$-\left(\frac{mv^2}{r}\right)dr = \frac{1}{2}m\left(v+dv\right)^2 - \frac{1}{2}mv^2 \tag{4.18}$$

$$-\left(\frac{mv^2}{r}\right)dr = \frac{1}{2}m\left(\cancel{v^2} + 2vdv + \cancel{dv^2} - \cancel{v^2}\right) \tag{4.19}$$

which yields:

$$
\begin{aligned}
-\left(\frac{\cancel{m}v^2}{r}\right)dr &= \frac{1}{\cancel{2}}\cancel{m}\left(\cancel{2}vdv\right) \\
-\frac{dr}{r} &= \frac{dv}{v} \\
d\left(\ln v + \ln r\right) &= 0 \\
\ln v + \ln r &= \text{constant} \\
vr &= \text{constant} \\
mvr &= \text{constant}'
\end{aligned}
$$

As we can see in this particular case the magnitude of the angular momentum ($L = mrv$) is conserved. It can be easily deduced that in central force field (like gravitational) the angular momentum is constant of the motion.

$$\boldsymbol{L} = m\boldsymbol{r} \times \boldsymbol{v} = \boldsymbol{r} \times m\boldsymbol{v} = \boldsymbol{r} \times \boldsymbol{p} \tag{4.20}$$

We can use the product rule, which works with the cross product:

$$
\begin{aligned}
\frac{d}{dt}\left(\boldsymbol{r} \times \boldsymbol{p}\right) &= \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} \\
\frac{d}{dt}\left(\boldsymbol{r} \times \boldsymbol{p}\right) &= \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \boldsymbol{F} \\
\frac{d}{dt}\left(\boldsymbol{r} \times \boldsymbol{p}\right) &= \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{F}
\end{aligned}
$$

where $\boldsymbol{v} \times m\boldsymbol{v}$ is identically zero. However, in central force field the force is parallel with the $\boldsymbol{r}$ vector, hence $\boldsymbol{r} \times \boldsymbol{F}$ is zero as well.

$$
\begin{aligned}
\frac{d}{dt}\boldsymbol{L} &= 0 \\
\boldsymbol{L} &= \text{constant}
\end{aligned}
$$

Since gravity is a central force, the angular momentum is constant. Assume that a small body in space orbits a large one (like a planet around the sun) along an elliptical path, with the large body being located at one of the ellipse foci. At the closest (Perihelion) and furthest (Aphelion) approaches, the angular momentum is perpendicular to the distance from the mass orbited, therefore:

$$L = mrv = \text{constants}$$

so in case if r is small (Perihelion) v must be large and vice versa.

4.1.7 Problems connected with rolling objects

A mechanical system consists of a massive pulley of rotational I, radius R, and mass m with a block (m) hanging from the end of a cord as it can be seen in figure (4.8)).

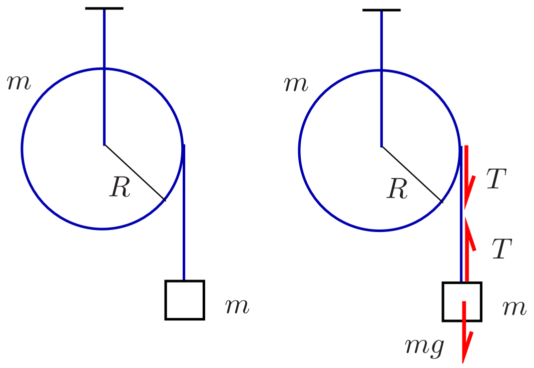


Figure 4.8: Rotation around fixed axis.

Assume that the pulley is free to turn without friction and the cord does not slip. Ignore air resistance. If the mass is released, analyse the motion of the system according to Newton's second law for linear and rotational motion

$$mg - T = m\,a \tag{1}$$
$$TR = I\alpha \tag{2}$$

The constraint relation between the linear and circular motion reads as:

$$a = R\alpha \tag{3}$$

We get two equations with two unknowns

$$mg - T = m\,a \tag{1}$$
$$T\cancel{R} = \frac{1}{2}m\cancel{R^2}\frac{a}{\cancel{R}} \tag{2}$$

We can write down the equations as a simplified form

$$mg - T = m\,a \tag{1'}$$
$$T = \frac{1}{2}m\,a \tag{2'}$$

By adding the two equations we can eliminate T

$$\cancel{m}g = \frac{3}{2}\cancel{m}\,a \quad ((1') + (2'))$$
$$a = \frac{2g}{3}; \quad \alpha = \frac{2g}{3R}; \quad T = \frac{mg}{3}$$

4.1.8 Kinematics of a Rolling Object

An elementary displacement of a point of a rigid body $d\boldsymbol{s}$ can be resolved into two displacements—the "translational" one $d\boldsymbol{s}_t$ and "rotational" one $d\boldsymbol{s}_r$:

$$d\boldsymbol{s} = d\boldsymbol{s}_t + d\boldsymbol{s}_r$$

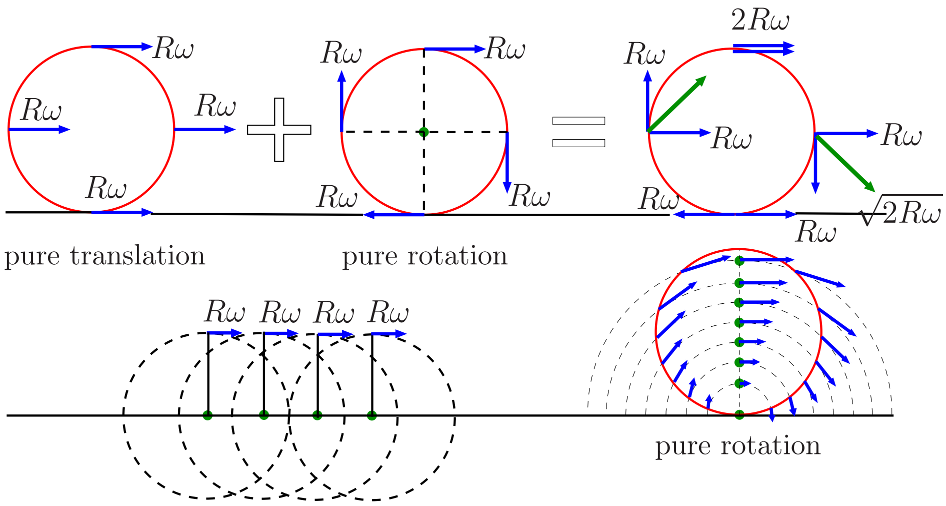


Figure 4.9: Slipless rotation around the axis of symmetry (above), and around the instantaneous axis (at the bottom)

where $d\boldsymbol{s}_t$ is the same for all the points of the body. This resolution of the displacement $d\boldsymbol{s}$ can be performed in different ways. When an object is rolling on a plane without slipping (see figure (4.9)), the point of contact of the object with the plane does not move. If we imagine a wheel moving forward by rolling on a plane at speed v, it must also be rotating about its axis at an angular speed ω since it is rolling. The object's angular velocity ω is directly proportional to its velocity, because as we know, the faster we are driving in a car, the faster the wheels have to turn. So, to determine the exact relationship between linear velocity and angular velocity, we can imagine the scenario in which the wheel travels a distance of x while rotating through an angle θ.

In mathematical terms, the length of the arc is equal to the angle of the segment multiplied by the object's radius, R. Therefore, we can say that the length of the arc of the wheel that has rotated an angle θ, is equal to $R\theta$. Furthermore, since the wheel is in constant contact with the ground, the length of the arc correlating to the angle is also equal to x. Therefore,

$$x = R\theta$$

We can take the derivative of both sides to obtain:

$$\frac{dx}{dt} = R\frac{d\theta}{dt} \tag{4.21}$$

$$v = R\omega \tag{4.22}$$

A rolling object combines translational motion of the centre of mass with rotation about an axis that passes through the centre of mass. For an object that is rolling without slipping, $v_{\text{CM}} = R\omega$.

The instantaneous center of velocity (IC) is a unique reference point which momentarily has a velocity of zero magnitude. Thus, as far as velocities are concerned, the body seems to rotate about the instantaneous center, that is, the velocity of any point on the rigid body is simply the angular velocity of the rigid body times the distance to the IC. Consequently, plane motion of a rigid body can be considered as a number of consecutive elementary rotations about instantaneous axes.

4.1.9 Dynamics of a Rolling Cylinder

Let us calculate the static friction force exerted on a solid cylinder of mass m, and radius of R is accelerating with parallel force to the ground (see figure (4.10)). The pulling force is assumed to be constant. What is the acceleration of the central mass?

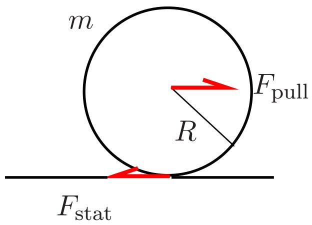


Figure 4.10: A cylinder is accelerated by a constant pulling force.

We can apply Newton's second law for linear and rotational motion as follows:

$$F_{\text{pull}} - F_{\text{stat}} = m\,a \tag{1}$$
$$F_{\text{stat}}\,R = I\alpha \tag{2}$$
$$a = R\alpha \tag{3}$$

Substituting the moment of inertia of the cylinder and taking into account the constraint relation the equations are the following:

$$F_{\text{pull}} - F_{\text{stat}} = m\,a \tag{1}$$
$$F_{\text{stat}}\,\cancel{R} = \frac{1}{2} m\,\cancel{R^2}\,\frac{a}{\cancel{R}} \tag{2'}$$

$$F_{\text{pull}} - F_{\text{stat}} = m\,a \tag{1}$$
$$F_{\text{stat}} = \frac{1}{2} ma \tag{2''}$$

By adding the two equations we can eliminate F_{stat}

$$F_{\text{pull}} = \frac{3}{2} m\,a \quad (1) + (2'')$$
$$a = \frac{2F_{\text{pull}}}{3m} \quad \text{acceleration}$$
$$F_{\text{stat}} = \frac{1}{\cancel{2}} \cancel{m} \frac{\cancel{2}F_{\text{pull}}}{3\cancel{m}} = \frac{F_{\text{pull}}}{3} \quad \text{static friction force}$$

4.1.10 Dynamics of a Rolling Spool

As it can be seen in figure (4.11) a spool of wire of mass m, and radius R is unwound under a constant force F_{pull}. Assuming the spool is a uniform solid cylinder that rolls without slipping. Find the acceleration of the spool's centre of mass. According to

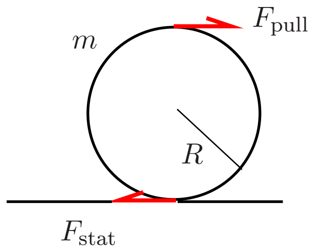


Figure 4.11: A spool is accelerated by a constant pulling force.

Newton's second law for linear and rotational motion:

$$F_{\text{pull}} - F_{\text{stat}} = m\,a \tag{1}$$
$$F_{\text{pull}}R + F_{\text{stat}}\,R = I\alpha \tag{2}$$

The constraint relation between the linear and circular motion reads as:

$$a = R\alpha \tag{3}$$

We can write down the equations as a simplified form:

$$F_{\text{pull}} - F_{\text{stat}} = m\,a \tag{1}$$
$$F_{\text{pull}}\cancel{R} + F_{\text{stat}}\,\cancel{R} = \frac{1}{2}m\,\cancel{R^2}\frac{a}{\cancel{R}} \tag{2}$$

After the simplification the equations take the following forms:

$$F_{\text{pull}} - F_{\text{stat}} = m\,a \tag{1}$$
$$F_{\text{pull}} + F_{\text{stat}} = \frac{1}{2}ma \tag{2}$$

By adding the equations we can eliminate the friction force F_{stat}

$$2F_{\text{pull}} = \frac{3}{2}m\,a \quad (1)+(2)$$
$$a = \frac{4F_{\text{pull}}}{3m} \quad \text{acceleration}$$

By subtracting the equations we can eliminate F_{pull}

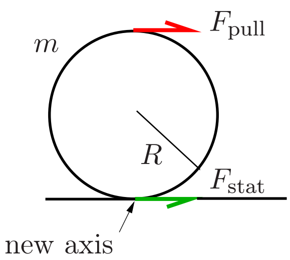
$$
\begin{aligned}
F_{\text{pull}} - F_{\text{stat}} - F_{\text{pull}} - F_{\text{stat}} &= m\,a - \frac{1}{2}ma \quad \text{(1)-(2)} \\
\cancel{F_{\text{pull}}} - F_{\text{stat}} - \cancel{F_{\text{pull}}} - F_{\text{stat}} &= \frac{1}{2}ma \\
-2F_{\text{stat}} &= \frac{m\,a}{2} \\
-F_{\text{stat}} &= \frac{\cancel{m}}{\cancel{4}}\frac{\cancel{4}F_{\text{pull}}}{3\cancel{m}} \\
F_{\text{stat}} &= -\frac{F_{\text{pull}}}{3} \quad \text{static friction force}
\end{aligned}
$$

4.1.11 Instant centre of rotation

As we mentioned earlier the instant centre of rotation, also called instantaneous velocity centre, is the point fixed to a body undergoing planar movement that has zero velocity at a particular instant of time. If we choose the instant centre of rotation as a rotation axis we can eliminate the static friction force. However, the moment of inertia with respect to the symmetry axis of the cylinder is no more valid. According to the parallel-axis theorem the new moment of inertia $I^\star$ reads as:

$$I^\star = I + mb^2 = \left(\frac{1}{2}mR^2 + mR^2\right) = \frac{3}{2}mR^2$$

The special choice of the axis allows us to calculate α using only one equation



$$
\begin{aligned}
\tau &= I^\star\alpha \\
F_{\text{pull}}\,2R &= \frac{3}{2}mR^2\alpha \\
\alpha &= \frac{4F_{\text{pull}}}{3mR} \qquad a = \frac{4F_{\text{pull}}}{3m} \checkmark
\end{aligned}
$$

4.1.12 Slipless rotation without static friction force

Let us consider the following problem associated with rolling without slipping. A cylinder is rolling without slipping in a horizontally-oriented surface. We assume that the angular acceleration α is constant during the motion. If the pulling force is constant and parallel with the surface of the ground, the direction and magnitude of the static friction force exerted by the surface of the ground can be obtained from the following equations:

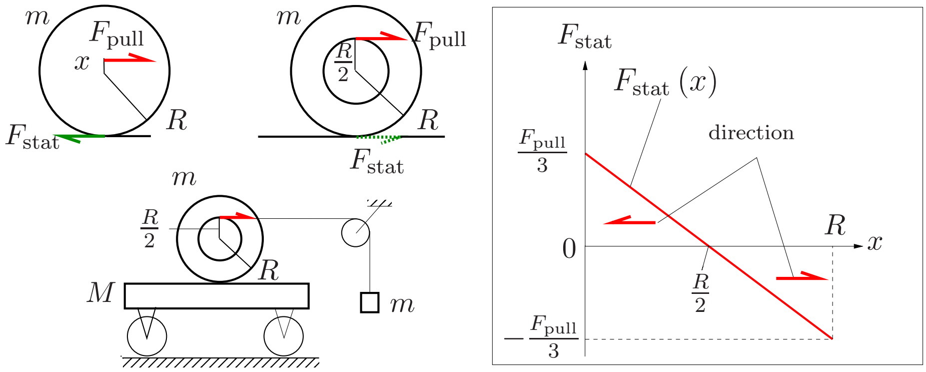


Figure 4.12: The static friction can be eliminated by the proper choice of the arm lever of the pulling force.

$$F_{\text{pull}} - F_{\text{stat}} = ma_{\text{CM}} \tag{4.23}$$

$$F_{\text{pull}}(R + x) = \frac{3}{2} mR^2 \alpha \tag{4.24}$$

We will drop the "CM" subscript of the acceleration ($a_{\text{CM}} = a$) and taking into account the constraint relation $\alpha = a/R$

$$F_{\text{pull}} - F_{\text{stat}} = ma \tag{4.25}$$

$$F_{\text{pull}}(R + x) = \frac{3}{2} mR^{\cancel{2}} \frac{a}{\cancel{R}} \tag{4.26}$$

$$\begin{aligned}
F_{\text{pull}}(R + x) &= \frac{3}{2} mRa \\
F_{\text{pull}}(R + x) &= \frac{3}{2} mR \left(\frac{F_{\text{pull}}}{m} - \frac{F_{\text{stat}}}{m} \right) \\
F_{\text{pull}}(R + x) &= \frac{3}{2} RF_{\text{pull}} - \frac{3}{2} RF_{\text{stat}} \\
\frac{3}{2} RF_{\text{stat}} &= \frac{3}{2} RF_{\text{pull}} - F_{\text{pull}}(R + x) \\
F_{\text{stat}} &= F_{\text{pull}} - \frac{2}{3} F_{\text{pull}} - \frac{2F_{\text{pull}}}{3R} x
\end{aligned}$$

we can see that static the static friction force depends linearly on the lever arm x

$$F_{\text{stat}}(x) = \frac{F_{\text{pull}}}{3} - \frac{2F_{\text{pull}}}{3R} x \tag{4.27}$$

$$F_{\text{stat}}(x) = \frac{F_{\text{pull}}}{3} \left(1 - \frac{2}{R} x \right) \tag{4.28}$$

According to equation (4.28) by a proper choice of the lever arm $x = R/2$ we can avoid slipping during the rotation even on an extremely slick surface. In this case there is no static friction force needed.

For this reason the acceleration of the massive platform in figure (4.12) will be zero. We can check it based on the dynamic equations of the rotational and linear motion. We have chosen the instantaneous axis of rotation. This axis is parallel with the axes of symmetry of the cylinder. Hence for $I^\star$ the Steiner's theorem is used.

$$\begin{aligned}
\tau &= I^* \alpha \\
F_{\text{pull}} \frac{3}{2} R &= I^* \alpha \\
F_{\text{pull}} \frac{3}{2} R &= \left(\frac{1}{2} m R^2 + m R^2 \right) \alpha \\
F_{\text{pull}} \cancel{\frac{3}{2}} \cancel{R} &= \left(\cancel{\frac{3}{2}} m R^{\cancel{2}} \right) \alpha \\
\alpha &= \frac{F_{\text{pull}}}{mR} \\
a &= \frac{F_{\text{pull}}}{m}
\end{aligned}$$

We assume that a static force exists and oriented from left to right. Let us calculate the magnitude of the static friction force F_{static}

$$\begin{aligned}
F_{\text{pull}} + F_{\text{static}} &= m\, a \\
F_{\text{static}} &= m\, a - F_{\text{pull}} \\
F_{\text{static}} &= \cancel{m} \frac{F_{\text{pull}}}{\cancel{m}} - F_{\text{pull}} \\
F_{\text{static}} = F_{\text{pull}} - F_{\text{pull}} &= 0 \checkmark
\end{aligned}$$

Obtaining a and α using work and energy

It is quite useful to obtain the solution of a given problem in several different ways. The following algorithm helps us to reproduce our former result based on the work-energy theorem and the conservation of energy as follows:

Work-Energy Theorem	Conservation of Energy
$W_{\text{rot}} = \frac{1}{2}I^\star{\omega_2}^2 - \frac{1}{2}I^\star{\omega_1}^2$	$(K + K_{\text{rot}} + V = \text{const})$
$\tau\,\varphi = \frac{1}{2}I^\star\omega^2$	$\frac{1}{2}I^\star\omega^2 = V_{\text{pull}}$
$F_{\text{pull}}\,\frac{3}{2}R\,\varphi = \frac{1}{2}I^\star\omega^2$	$\frac{1}{2}\frac{3}{2}m\,R^2\omega^2 = F_{\text{pull}}\,\frac{3}{2}R\varphi$
$F_{\text{pull}}\,\cancel{\frac{3}{2}}\cancel{R}\,\frac{x}{\cancel{R}} = \frac{1}{2}\cancel{\frac{3}{2}}mR^2\omega^2$	$\frac{1}{2}\cancel{\frac{3}{2}}m\,v^2 = F_{\text{pull}}\,\cancel{\frac{3}{2}}\,x$
$F_{\text{pull}}\,x = \frac{1}{2}mR^2\omega^2$	$\frac{1}{2}m\,v^2 = F_{\text{pull}}\,x$
$\omega^2 = \frac{2F_{\text{pull}}x}{mR^2}$	$v^2 = \frac{2F_{\text{pull}}\,x}{m}$
$\omega^2 = 2\alpha\frac{x}{R}$	$v^2 = 2a\,x$
$\frac{\cancel{2}F_{\text{pull}}\cancel{x}}{mR^{\cancel{2}}} = \cancel{2}\alpha\frac{\cancel{x}}{\cancel{R}}$	$\frac{\cancel{2}F_{\text{pull}}\,\cancel{x}}{m} = \cancel{2}a\,\cancel{x}$
$\alpha = \frac{F_{\text{pull}}}{mR}$	$a = \frac{F_{\text{pull}}}{m}$
$\frac{a}{\cancel{R}} = \frac{F_{\text{pull}}}{m\cancel{R}}$	

Wheel Rolling Without Slipping Down an Inclined Plane

A cylinder of radius R, mass m, and moment of inertia I about the axis passing through its centre of mass starts from rest and moves down an inclined plane at an angle θ from the horizontal (see figure (4.14)). The centre of mass of the cylinder has dropped a vertical distance when it reaches the bottom of the incline. The cylinder rolls down the incline without slipping. What is the acceleration of the central mass?

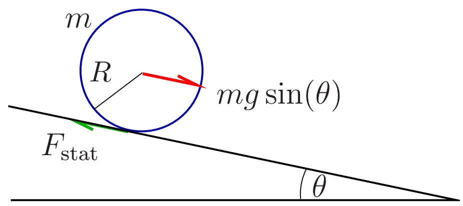


Figure 4.13: Rolling without slipping.

If we use the instantaneous axis of rotation, the problem becomes very straightforward

$$\tau = I^\star\alpha \tag{4.29}$$

$$\cancel{mg}\sin(\theta)\cancel{R} = \frac{3}{2}\cancel{m}R^{\cancel{2}}\frac{a}{\cancel{R}} \tag{4.30}$$

$$a = \frac{2g\sin(\theta)}{3} \tag{4.31}$$

$$
\begin{aligned}
mg\sin(\theta) - F_{\text{stat}} &= m\,a \\
F_{\text{stat}} &= mg\sin(\theta) - \frac{2mg\sin(\theta)}{3} \\
F_{\text{stat}} &= \frac{mg\sin(\theta)}{3}
\end{aligned}
$$

Now that we know the value of the static force of friction needed for rolling down of the cylinder without slipping, we can find the condition at which this rolling is possible. For the cylinder to roll down without slipping, the force $mg\sin(\theta)/3$ must not exceed the maximum value of the static force of friction

$$
\begin{aligned}
\mu \cancel{mg}\cos(\theta) &\geq \frac{\cancel{mg}\sin(\theta)}{3} \\
3\mu &\geq \tan(\theta)
\end{aligned}
$$

Consequently, if the slope of the plane exceeds the triple value of the static coefficient of friction between the cylinder and the plane, rolling down cannot occur without slipping.

4.1.13 Rolling and Skidding

A circular object is said to rolling and skidding when the velocity v of its centre of mass (or axis of rotation in this case) is not equal to $R\omega$. The point in contact with the ground is no longer the instantaneous centre of rotation. The object can rotate faster or slower than it should do for pure rolling. The point in contact with the ground will have the tendency to move backwards due to greater angular velocity. Friction on this point will act in the forward direction and try to equalise $R\omega$ and v. This happens to the rear wheel of a sport bike when tried to accelerate fast from rest. The smoke is the result of heat due to friction. In this case the object is rotating slower. The point in contact will have the tendency to move forward.

In an attempt to equalise $R\omega$ and v friction will act in backward direction. This happens when a bowling ball strikes the floor. When it is released from hand it does not spin so fast. So when it comes in contact with the floor it slips for an instance before undergoing pure rotation.

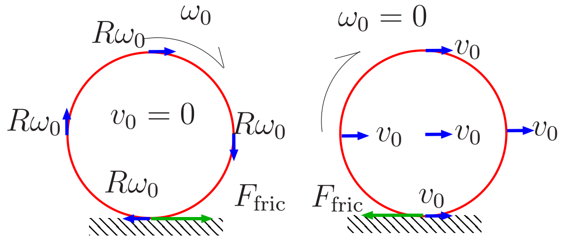


Figure 4.14: Two cylinder skidding during rotation. The initial conditions are different.

In case the initial velocity of the central mass is zero the change of speed of the lowest point can be written as:

$$v(t) = R\omega_0 - R\alpha t - v_{\mathrm{CM}}(t)$$

The α and a_{CM} during skidding:

$$\begin{aligned} \tau &= I\alpha & F &= ma_{\mathrm{CM}} \\ F_{\mathrm{fric}}R &= \tfrac{1}{2}mR^2\alpha & F_{\mathrm{fric}} &= ma_{\mathrm{CM}} \\ \mu mg\cancel{R} &= \tfrac{1}{2}mR^{\cancel{2}}\alpha & \mu\cancel{m}g &= \cancel{m}a_{\mathrm{CM}} \\ \alpha &= \tfrac{2\mu g}{R} & a_{\mathrm{CM}} &= \mu g \end{aligned}$$

$$\begin{aligned} v(t) &= R\omega_0 - R\frac{2\mu g}{R}t - a_{\mathrm{CM}}\,t \\ v(t) &= R\omega_0 - \cancel{R}\frac{2\mu g}{\cancel{R}}t - \mu g\,t \\ v(t) &= R\omega_0 - 3\mu g t \end{aligned}$$

where α stands for the angular acceleration of the cylinder. In that moment (t') the speed of the lowest point becomes zero (compared to the ground) and skidding stopes.

$$\begin{aligned} 0 &= R\omega_0 - 3\mu g t \\ t' &= \frac{R\omega_0}{3\mu g} \end{aligned}$$

The velocity of the central mass at that moment:

$$v_{\mathrm{CM}} = a_{\mathrm{CM}}t' = \cancel{\mu g}\frac{R\omega_0}{3\cancel{\mu g}} = \frac{R\omega_0}{3}$$

The displacement of the central mass during the skidding:

$$X = \frac{\left(\cancelto{0}{v_0} + v_{\mathrm{CM}}\right)t'}{2} = \frac{1}{2}v_{\mathrm{CM}}t' = \frac{R^2\omega_0^2}{18\mu g}$$

In case the initial angular velocity is zero, but the speed of the central mass is v_0 the change of speed of the lowest point can be written as follows:

$$v(t) = v_{\mathrm{CM}} - R\omega(t)$$

It should be noted that the magnitudes of a and α are the same as before.

$$\begin{aligned} v(t) &= v_0 - \mu g t - R\alpha t \\ v(t) &= v_0 - \mu g t - R\frac{2\mu g}{R}t \\ v(t) &= v_0 - 3\mu g t \end{aligned}$$

where α stands for the angular acceleration of the cylinder. In that moment (t'') the speed of the lowest point becomes zero (compared to the ground) and skidding stops.

$$0 = v_0 - 3\mu g t$$
$$t'' = \frac{v_0}{3\mu g}$$

The velocity of the central mass at that moment:

$$v_0 - a_{\mathrm{CM}} t'' = v_0 - \cancel{\mu g}\frac{v_0}{3\cancel{\mu g}} = \frac{2v_0}{3}$$

The displacement of the central mass during the skidding

$$X' = \frac{v_1 + v_2}{2} t'' = \left(\frac{3}{3}v_0 + \frac{2}{3}v_0\right)\frac{1}{2}\frac{v_0}{3\mu g} = \frac{5R^2\omega_0^2}{18\mu g} = \frac{5v_0^2}{18\mu g}$$

If the initial energies of the cylinders are the same

$$\frac{1}{2}\frac{1}{2}mR^2\omega_0^2 = \frac{1}{2}mv_0^2$$
$$\omega_0^2 = \frac{2v_0^2}{R^2}$$

we can compare the displacements of the cylinders. The ratio of the skidding displacements is

$$\frac{X'}{X} = \frac{5v_0^2}{18\mu g} / \frac{2v_0^2}{18\mu g} = \frac{5}{2} = 2.5$$

Chapter 5

Oscillations

Oscillations are defined as processes distinguished by a certain degree of repetition. For example, the swings of a clock pendulum, the vibrations of a string or the voltage across the plates of a capacitor in a radio receiver circuit have this property of repetition. Depending on the physical nature of the repetiting process, we distinguish mechanical, electromagnetic, sound, and other oscillations. Oscillations are widespread in nature and engineering. Depending on the nature of action on an oscillating system, we distinguish free (or natural) oscillations, forced oscillation, auto-oscillation, and parametric oscillations.

Free or *natural oscillations* occur in a system left alone after an impetus was imparted to it or it was brought out of the equilibrium position. An example is the oscillations of a ball suspended on a string. To initiate oscillations, we may either push the ball or move it to the side and release it.

In *forced oscillations*, the oscillating system is acted upon by an external periodically changing force. An example here is the oscillations of a bridge caused by people walking over it. It means an organized, uniformed, steady and rhythmic walking.

Auto-oscillations, like forced ones, are attached by the action of external forces on the oscillating system, but the moments of time when these oscillations are exerted are set by the oscillating system itself—the latter controls the external action. Examples of an auto oscillating system are clocks in which a pendulum receives pushes at the expense of the energy of a lifted weight or a coiled spring, and these pushes occur when the pendulum passes through its middle position.

In *parametric oscillations*, external action causes periodic changes in a parameter of a system, for instance, in the length of a thread on which an oscillating ball is suspended.

Harmonic oscillations are the simplest ones. These are oscillations when the oscillating quantity changes with time according to a sine or cosine law.

5.1 Simple Harmonic Motion

The simple harmonic motion (SHM) is especially important for the following reason: oscillations in nature and engineering are often close to harmonic ones in their character, and second, periodic processes of a different form can be represented as the superposition of several harmonic oscillations (*Fourier-analysis*).

Let us consider oscillations described by the equation:

$$\ddot{x} + \omega^2 x = 0$$

Such oscillations are performed by a body of mass m experiencing typically the elastic force $F = -kx$. The general solution of equation 5.1 is:

$$x(t) = A\cos(\omega t + \gamma) \tag{5.1}$$

where $\omega = \sqrt{k/m}$, $A = x_{\text{max}}$ and γ are arbitrary constants.

The energy equation of a linear oscillator:

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{5.2}$$

$$\omega^2 = \frac{k}{m} \tag{5.3}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{5.4}$$

The energy of many oscillatory systems can be described in the following form:

$$E_{\text{tot}} = \frac{1}{2}\alpha\dot{q}^2 + \frac{1}{2}\beta q^2$$

$$\omega^2 = \frac{\beta}{\alpha} \to T = 2\pi\sqrt{\frac{\alpha}{\beta}}$$

where q is a general physical quantity (charge, displacement, current) and α, β are constants.

Let us look at some examples. An ideal LC-circuit is an oscillatory system (see figure 5.1).

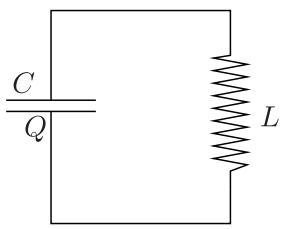


Figure 5.1: An LC-circuit.

$$
\begin{aligned}
E_{\text{tot}} &= E_{\text{magnetic}} + E_{\text{electric}} \\
E_{\text{tot}} &= \frac{1}{2}LI^2 + \frac{1}{2}CU^2 \quad /I = \dot{Q};\ U = \frac{Q}{C}/ \\
E_{\text{tot}} &= \frac{1}{2}L\dot{Q}^2 + \frac{1}{2}\frac{1}{C}Q^2 \ \ /\alpha = L;\ \beta = \frac{1}{C}/ \\
\omega^2 &= \frac{\beta}{\alpha} = \frac{1}{LC} \\
\omega &= \frac{1}{\sqrt{LC}}
\end{aligned}
$$

Our second example is a "U-tube" in the Earth's gravitational field. A U-tube, as it can be seen in figure (5.2) open at both ends is filled with an incompressible fluid of density ρ. The cross-sectional area A of the tube is uniform and the total length of the fluid in the tube is l. A piston is used to depress the height of the liquid column on one side by a distance x, (raising the other side by the same distance) and then is quickly removed. What is the angular frequency of the ensuing simple harmonic motion? Neglect any resistive forces and at the walls of the U-tube. The restoring

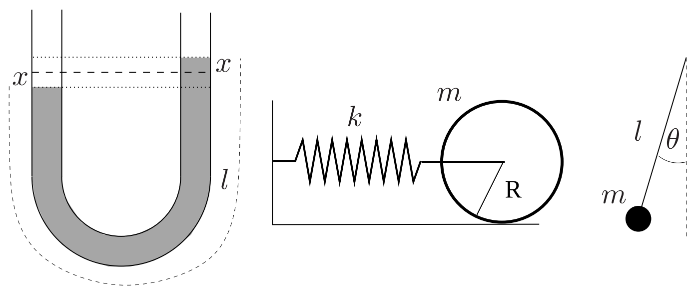


Figure 5.2: U-tube, rolling oscillator, and a simple pendulum.

force is proportional to the gravitational force of the displaced mass element $\Delta m\, g$ and the whole liquid column (m) is oscillating, hence:

$$
\begin{aligned}
-\Delta m\, g &= ma \\
-\cancel{\rho A}2xg &= \cancel{\rho A}l\,\ddot{x} \\
\ddot{x} + \frac{2g}{l}x &= 0 \\
\omega &= \sqrt{\frac{2g}{l}}
\end{aligned}
$$

As a third example consider a cylinder which rolls without slipping in a horizontal surface and is held by a spring. The central mass of the cylinder performs simple harmonic motion. The other points move in a more complex way. The total energy of the system is given by

$$E_{\text{tot}} = \frac{1}{2}I^{\star}\dot{\varphi}^2 + \frac{1}{2}kx^2$$
$$E_{\text{tot}} = \frac{1}{2}\frac{3}{2}mR^2\dot{\varphi}^2 + \frac{1}{2}kR^2\varphi^2$$
$$\omega^2 = \frac{\beta}{\alpha} = \frac{k\cancel{R^2}}{\frac{3}{2}m\cancel{R^2}} = \frac{2k}{3m}$$
$$\omega = \sqrt{\frac{2k}{3m}}$$

As a fourth example, we will study the oscillations of a pendulum. The pendulum to be a rigid body performing oscillations about a fixed point or axis under the action of the force of gravity. A simple pendulum is defined as an idealized system consisting of a weightless and undeformable rod on which a mass concentrated at one point is suspended. The energy expression take the following form:

$$E_{\text{tot}} = mgl\,(1 - \cos(\theta)) + \frac{1}{2}ml^2\dot{\theta}^2$$

As we can see this equation cannot be written down in the following form:

$$E_{\text{tot}} = \frac{1}{2}\alpha\dot{q}^2 + \frac{1}{2}\beta q^2$$

In conclusion we can state that in general the pendulum oscillates anharmonically. However, if we restricted only small-amplitude oscillations, we can thus assume that cosine function can be well approximated by the first two terms of their power series

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2!}$$

Hence:

$$E_{\text{tot}} = mgl\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right) + \frac{1}{2}ml^2\dot{\theta}^2$$
$$E_{\text{tot}} = \frac{1}{2}mgl\theta^2 + \frac{1}{2}ml^2\dot{\theta}^2$$
$$\omega^2 = \frac{\cancel{m}g\cancel{l}}{\cancel{m}l^{\cancel{2}}} = \frac{g}{l}$$
$$\omega = \sqrt{\frac{g}{l}}$$

If the angular displacement is so small that $\sin(\theta) \approx \theta$, the equation of motion of a simple pendulum takes the following form:

$$-\cancel{m}g\sin\theta = \cancel{m}l\ddot{\theta}$$
$$-g\theta = l\ddot{\theta}$$
$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Strictly speaking the simple pendulum never oscillates harmonically. As θ decreases the deviation from SHM can be arbitrarily small.

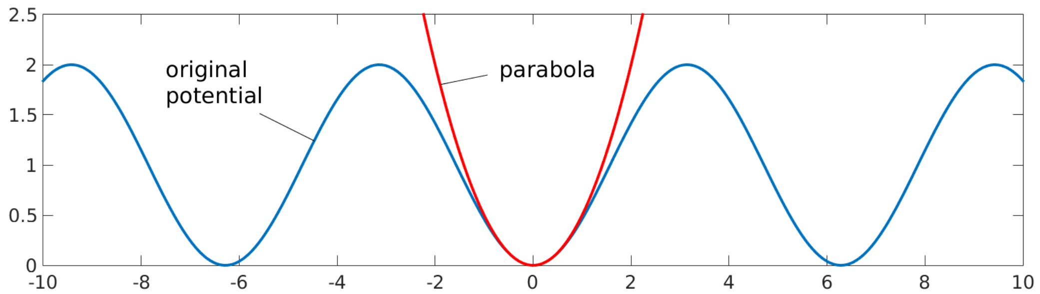


Figure 5.3: In case the θ is small the potential function can be well approximated by $mgl\frac{\theta^2}{2!}$

5.1.1 Oscillation of a Disk inside a Semicircular Well

A cylinder of mass m and radius r rolls without slipping inside a semicircular track of radius R that occupies a vertical plane as it can be seen in figure (5.4). Let φ represent the angle of rotation of the cylinder, and let ϕ measure the angle between the centre of the semicircle and the centre of the cylinder, with respect to the vertical. Calculate the period of small oscillations about the bottom of the track by considering small displacements from equilibrium.

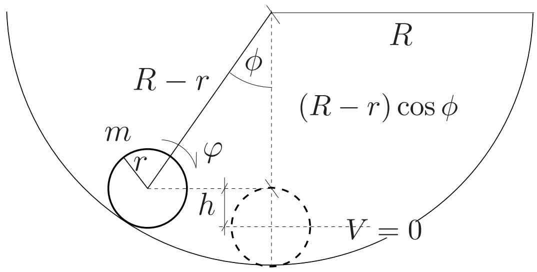


Figure 5.4: A disk rolls back and forth in a semicircular well.

The total energy of the system can be expressed in the following form:

$$E_{\text{tot}} = mgh + \frac{1}{2}I^{\star}\dot{\varphi}^2$$

$$E_{\text{tot}} = mg\left(R-r\right)\left(1-\cos\left(\phi\right)\right) + \frac{1}{2}\frac{3}{2}mr^2\dot{\varphi}^2$$

Due to the constraint relation the velocity of the central mass of the cylinder is $v = r\dot{\varphi}$. Since the central mass of the body performs circular motions v can be expressed by $\dot{\phi}$ as well and the two velocities must be the same

$$\left(R-r\right)\dot{\phi} = r\dot{\varphi}$$

Hence the total energy takes the following form:

$$E_{\text{tot}} = mg\left(R-r\right)\left(1-\cos\left(\phi\right)\right) + \frac{1}{2}\frac{3}{2}m\left(R-r\right)^2\dot{\phi}^2$$

As we can see this equation (similar to our previous problem) cannot be written down in the form:

$$E_{\text{tot}} = \frac{1}{2}\alpha\dot{q}^2 + \frac{1}{2}\beta q^2$$

If we restricted only small-amplitude oscillations, we can thus assume that cosine function can be well approximated by the first two terms of their power series

$$\cos(\phi) \approx 1 - \frac{\phi^2}{2!}$$

Hence the energy expression in small angle approximation is as follows:

$$E_{\text{tot}} = mg\left(R-r\right)\left(\cancel{1} - \left(\cancel{1} - \frac{\phi^2}{2}\right)\right) + \frac{1}{2}\frac{3}{2}m\left(R-r\right)^2\dot{\phi}^2$$
$$E_{\text{tot}} = \frac{1}{2}mg\left(R-r\right)\phi^2 + \frac{1}{2}\frac{3}{2}m\left(R-r\right)^2\dot{\phi}^2$$

Finally, the time period T can easily be calculated

$$\omega = \sqrt{\frac{\beta}{\alpha}} = \sqrt{\frac{mg\left(R-r\right)}{\frac{3}{2}m\left(R-r\right)^2}} = \sqrt{\frac{2g}{3\left(R-r\right)}}$$
$$T = 2\pi\sqrt{\frac{3\left(R-r\right)}{2g}}$$

Further general description of this system will be discussed in the last chapter.

Spring-body system

Let us consider an ideal (massless) spring and a linked body (figure 5.5). At the initial time moment the spring is undistorted, and the velocity of the joined body is zero. At a given moment a constant force is applied to the body. The friction force is neglected. How far does the body go compared to the initial position?

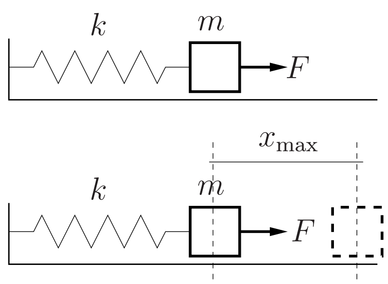


Figure 5.5: The spring-body system.

Let us solve this problem in the framework of oscillations. The equation of motion is the following:

$$F - kx = m\ddot{x} \tag{5.5}$$

After rearranging the terms we get

$$\begin{aligned} 0 &= \ddot{x} + \frac{k}{m}x - \frac{F}{m} \\ 0 &= \ddot{x} + \frac{k}{m}\left(x - \frac{F}{k}\right) \end{aligned}$$

By introducing a new coordinate $\chi = x - \frac{F}{k}$ the equation takes the following form:

$$\begin{aligned} 0 &= \ddot{\chi} + \frac{k}{m}\chi \\ \ddot{\chi} &= \ddot{x} \end{aligned}$$

It is a simple harmonic motion but the centre of the vibration is shifted by F/k. The body oscillates around the equilibrium position. Since the amplitude is F/k the maximum displacement from the initial position is as much as $2F/k$. The maximum speeds and accelerations are as follows:

$$v_{\text{max}} = \pm A\omega = \pm\frac{F}{k}\sqrt{\frac{k}{m}} = \pm\frac{F}{\sqrt{km}} \tag{5.6}$$

$$a_{\text{max}} = \pm A\omega^2 = \pm\frac{F}{k}\frac{k}{m} = \pm\frac{F}{m} \tag{5.7}$$

Normal-Modes of a Coupled Linear Oscillator

In accordance with figure (5.6) two masses m_1 and m_2 are joined by a spring of spring constant k. The masses can slide back and forth along a line. Show that the frequency of vibration of these masses along the line connecting them is:

$$\omega = \sqrt{k\frac{m_1 + m_2}{m_1 m_2}}$$

In a one-dimensional system at a given mode the vibration will have nodes, or places where the displacement is always zero. These nodes correspond to points in the mode shape where the mode shape is zero. Since the vibration of a system is given by the mode shape multiplied by a time function, the displacement of the node points remains zero at all times. The node of the system will be the central mass, the position of the CM by definition

$$X_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{5.8}$$

We will use new coordinates, relative to the CM as follows:

$$\begin{aligned} x_{12} &= X_{\text{CM}} - x_1 = \frac{m_2}{m_1 + m_2}(x_2 - x_1) \\ x_{22} &= x_2 - X_{\text{CM}} = \frac{m_1}{m_1 + m_2}(x_2 - x_1) \end{aligned}$$

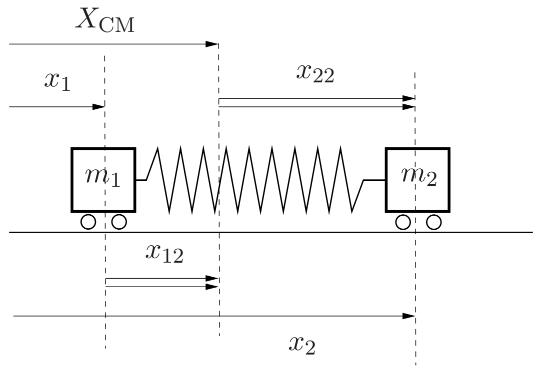


Figure 5.6: Notations of the coupled linear oscillator.

Introducing $x = (x_2 - x_1)$

$$x_{12} = X_{\mathrm{CM}} - x_1 = \frac{m_2}{m_1 + m_2} x$$
$$x_{22} = x_2 - X_{\mathrm{CM}} = \frac{m_1}{m_1 + m_2} x$$

The coupled oscillator can be treated as two independent oscillators. The equation of motion for the first and second oscillator reads as:

$$-k_1 x_{12} = m_1 \ddot{x}_{12} \tag{5.9}$$
$$-k_2 x_{22} = m_1 \ddot{x}_{22} \tag{5.10}$$

Two or more springs are said to be in series when they are connected end-to-end or point to point. The equivalent spring constant is:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \tag{5.11}$$

The ratio between the two parts of the spring is:

$$\frac{k_1}{k_2} = \frac{x_{12}}{x_{22}} = \frac{m_2}{m_1} \tag{5.12}$$

From equation (5.11) and (5.12) the k_1 and k_2 spring constants are:

$$k_1 = \frac{m_1 + m_2}{m_2} k$$
$$k_2 = \frac{m_1 + m_2}{m_1} k$$

Substituting k_1, k_2, x_{12} and x_{22} into equations (5.9), (5.10)

$$-\frac{\cancel{m_1 + m_2}}{\cancel{m_2}} k \frac{\cancel{m_2}}{\cancel{m_1 + m_2}} x = m_1 \frac{m_2}{m_1 + m_2} \ddot{x} \tag{5.13}$$
$$-\frac{\cancel{m_1 + m_2}}{\cancel{m_1}} k \frac{\cancel{m_1}}{\cancel{m_1 + m_2}} x = m_2 \frac{m_1}{m_1 + m_2} \ddot{x} \tag{5.14}$$

the equations (5.13), (5.14) become identical:

$$-kx = \frac{m_1 m_2}{m_1 + m_2}\ddot{x} \tag{5.15}$$

$$\ddot{x} + k\frac{m_1 + m_2}{m_1 m_2}x = 0 \tag{5.16}$$

$$\omega = \sqrt{k\frac{m_1 + m_2}{m_1 m_2}} \tag{5.17}$$

There is another way of solution. Let the amount of deformation of the spring at the initial time moment Δx_0 (it is assumed to be known). We can use the conservation of linear momentum and the conservation of mechanical energy as well. At the initial moment the total linear momentum is zero and the total energy of the system is stored as an elastic energy $(1/2k\Delta x_0^2)$. We expect that the bodies reach the maximum speed when the spring is undeformed.

$$0 = m_1 v_1 - m_2 v_2 \tag{5.18}$$

$$\frac{1}{2}k\Delta x_0^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \tag{5.19}$$

By solving the equations we get the maximum velocities:

$$v_1^2 = \frac{k\Delta x_0}{m_1 + \frac{m_1^2}{m_2}} \tag{5.20}$$

$$v_2^2 = \frac{k\Delta x_0}{m_2 + \frac{m_2^2}{m_1}} \tag{5.21}$$

These velocities can be expressed by the corresponding amplitudes as follows:

$$v_1 = A_1\omega \tag{5.22}$$

$$v_2 = A_2\omega \tag{5.23}$$

Although A_1 and A_2 are unknown $A_1 + A_2$ must be Δx_0. Hence

$$v_1 + v_2 = \Delta x_0\,\omega \tag{5.24}$$

The solution of this equation yields:

$$\omega = \frac{v_1 + v_2}{\Delta x_0} = \sqrt{k\frac{m_1 + m_2}{m_1 m_2}}$$

Normal-Modes of a Modified Coupled Oscillator

Let us consider a mechanical system which consists of a hoop and a cylinder (see figure 6.13). Each having mass m and radius R, are linked by an ideal spring. The bodies are free to roll without slipping. We compress the spring by a rope. At a given moment the rope is burnt. We assume that during the oscillation there will be a stationary point

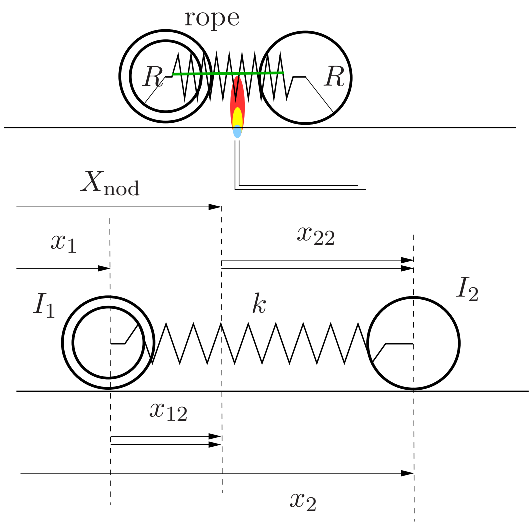


Figure 5.7: A coupled oscillator with rolling.

(node) of the system and the conservation of energy and angular momentum remains valid. At the initial moment the total energy of the system is stored as an elastic energy ($1/2k\Delta x^2$). Let us calculate the maximum velocities of the central masses of the rigid bodies. At that moment when the spring is undeformed the bodies gain maximum speed. According to the conservation of mechanical energy and angular momentum we get two equations:

$$\frac{1}{2}k\Delta x^2 = \frac{1}{2}I_1^{\star}\dot{\varphi}_1^2 + \frac{1}{2}I_2^{\star}\dot{\varphi}_2^2 \tag{1}$$

$$0 = I_1^{\star}\dot{\varphi}_1 + I_2^{\star}\dot{\varphi}_2 \tag{2}$$

By substituting the moments of inertia of the rigid bodies with respect to the instantaneous axis of rotation the equations take the following forms:

$$\frac{1}{2}k\Delta x^2 = mR^2\omega_1^2 + \frac{3}{4}mR^2\omega_2^2 \tag{1$'$}$$

$$0 = 2mR^2\omega_1 + \frac{3}{2}mR^2\omega_2 \tag{2$'$}$$

From equation ($2'$) the angular velocity of the hoop is:

$$\omega_1 = -\frac{3}{4}\omega_2 \tag{5.25}$$

By substituting ω_1 into equation ($1'$) we get:

$$\omega_2 = \frac{\Delta x}{R}\sqrt{\frac{8k}{21m}} \tag{5.26}$$

$$\omega_1 = -\frac{\Delta x}{R}\sqrt{\frac{3k}{14m}} \tag{5.27}$$

The stationary point of the system is not equal to the central mass.

$$X_{\text{nod}} = \frac{I_1^\star x_1 + I_2^\star x_2}{I_1^\star + I_2^\star} = \frac{4x_1 + 3x_2}{7}$$

The reason behind is that external forces (static friction) also act on the bodies. The coordinates corresponding to the stationary point of the system are:

$$x_{12} = X_{\text{nod}} - x_1 = \frac{3}{7}\Delta x$$
$$x_{22} = x_2 - X_{\text{nod}} = \frac{4}{7}\Delta x$$

This problem is analogous with the previous one. For this reason we can guess the equation of motion as follows:

$$-kx = \frac{m_1 m_2}{m_1 + m_2}\ddot{x} \quad \rightleftarrows \quad -k\varphi = \frac{I_1^\star I_2^\star}{I_1^\star + I_2^\star}\frac{1}{R^2}\ddot{\varphi}$$
$$0 = \ddot{x} + k\frac{m_1 + m_2}{m_1 m_2}x \quad \rightleftarrows \quad 0 = \ddot{\varphi} + k\frac{I_1^\star + I_2^\star}{I_1^\star I_2^\star}R^2\varphi$$
$$\omega = \sqrt{k\frac{m_1 + m_2}{m_1 m_2}} \quad \rightleftarrows \quad \Omega = \sqrt{k\frac{I_1^\star + I_2^\star}{I_1^\star I_2^\star}R^2}$$

$$\Omega = \sqrt{k\frac{I_1^\star + I_2^\star}{I_1^\star I_2^\star}R^2} = \sqrt{k\frac{\left(2mR^2 + \frac{3}{2}mR^2\right)R^2}{2mR^2 \cdot \frac{3}{2}mR^2}} = \sqrt{\frac{7k}{6m}}$$

For this reason the equation of motion is:

$$\ddot{x} + \frac{7k}{6m}x = 0$$

The maximum velocities of the bodies can be calculated by the expression $v_{\max} = A\Omega$, hence:

$$v_1 = x_{12}\,\Omega = \frac{3}{7}\Delta x\sqrt{\frac{7k}{6m}} = \Delta x\sqrt{\frac{3k}{14m}} \tag{5.28}$$
$$v_2 = x_{22}\,\Omega = \frac{4}{7}\Delta x\sqrt{\frac{7k}{6m}} = \Delta x\sqrt{\frac{8k}{21m}} \tag{5.29}$$

5.1.2 Simple Pendulum

A simple pendulum consists of a mass m suspended from a fixed point O by a massless taut string (or massless rod). The system is treated as a rigid one. When the mass m is displaced from the virtual equilibrium position, it moves back and forth in an arc of a circle as shown. Thus the motion of a pendulum is equivalent to a rotational motion in a vertical plane.

$$\tau_z = I_z\ddot{\theta} \tag{5.30}$$
$$-mg\sin\theta l = ml^2\ddot{\theta} \tag{5.31}$$

$\tau_z = -(mg\sin\theta)\,l$ the negative sign is taken because the torque acts in such a way as to decrease the angle θ. After rearranging equation (5.31) we get:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{5.32}$$

This equation is not so easy to solve. But if we assume the angular displacement θ to be very small, this is, $\theta \ll \pi/2$, then $\sin\theta \approx \theta$ equation (5.32) takes the following form:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{5.33}$$

which has the solution

$$\theta = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t + \phi\right) \tag{5.34}$$

θ_0 and ϕ are arbitrary constants that determine the amplitude and the phase of the oscillations from the initial conditions. The frequency f and the time period T are independent of the amplitude of the oscillations, provided the amplitude is small enough. That T is independent of the amplitude for small displacement makes the pendulum well suited for use in clocks to regulate the rate.

5.1.3 Physical Pendulum

A physical pendulum is simply a rigid object which swings freely about some pivot point. As it can be seen in figure 5.8 the physical pendulum may be compared with a simple pendulum, which consists of a small mass suspended by a (ideally massless) string. For the simple pendulum, the bob is assumed to be a point mass. For the rigid

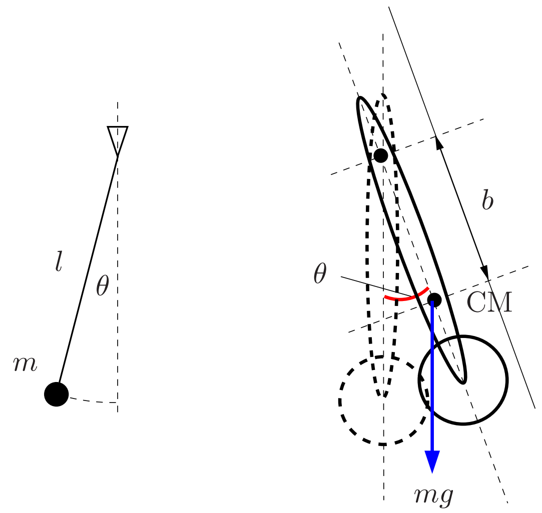


Figure 5.8: Simple and physical pendulum.

rod, however, the mass is uniformly distributed from the axis to a maximum distance L away from the axis.

To find the period of a physical pendulum, we first find the net torque acting on the physical pendulum and then use the rotational form of Newton's second law.

$$\begin{aligned}
\tau &= F_{\perp} r = -mg\sin(\theta)b \\
\tau &= -mg\,b\,\theta \\
-mg\,b\,\theta &= I\alpha \quad /\alpha = \ddot{\theta}\,/ \\
\ddot{\theta} &= -\frac{mgb}{I}\theta \\
0 &= \ddot{\theta} + \frac{mgb}{I}\theta \\
\omega = \sqrt{\frac{mgb}{I}} &\rightarrow T = 2\pi\sqrt{\frac{I}{mgb}}
\end{aligned}$$

The rotational inertia for of a uniform bar rotating about an axis through an endpoint is $I = \frac{1}{3}mL^2$. The period of oscillation is:

$$T = 2\pi\sqrt{\frac{2L}{3g}}$$

With small displacements from the equilibrium position, a physical pendulum performs harmonic oscillations whose frequency depends on the mass of the pendulum, the moment of inertia of the pendulum relative to the axis of rotation. A comparison shows that a mathematical pendulum of length

$$l_{\mathrm{r}} = \frac{I}{mb}$$

will have the same period of oscillations as the given physical pendulum. The quantity is called the reduced length of a physical pendulum. Thus, the reduced length of a physical pendulum is the length of a simple pendulum whose period of oscillations coincides with that of the given physical pendulum.

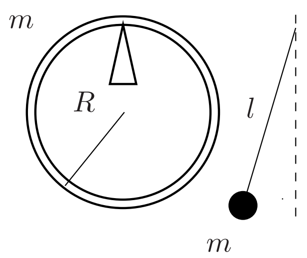


Figure 5.9: A hoop as physical pendulum and its oscillatory equivalent.

According to figure (5.9), a physical pendulum consists of a hoop suspended at its rim. What is the length l of a simple pendulum with the same period as the physical pendulum?

The central mass is at the centre of the hoop. The distance of the suspension from the CM is R. Hence, $b = R$ and the moment of inertia with respect to the suspension $I^\star = mR^2 + mR^2 = 2mR^2$. The reduced length of a mathematical pendulum is:

$$l_{\mathrm{r}} = \frac{I}{mb} = \frac{2\cancel{m}R^{\cancel{2}}}{\cancel{m}\cancel{R}} = 2R$$

5.1.4 Cycloidal Pendulum

A particle with mass m constrained to frictionless oscillation under gravity along the arc of a cycloid and having a period that is strictly independent of amplitude. A cycloid is the curve (figure 5.10) traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping. The detailed analysis of the problem

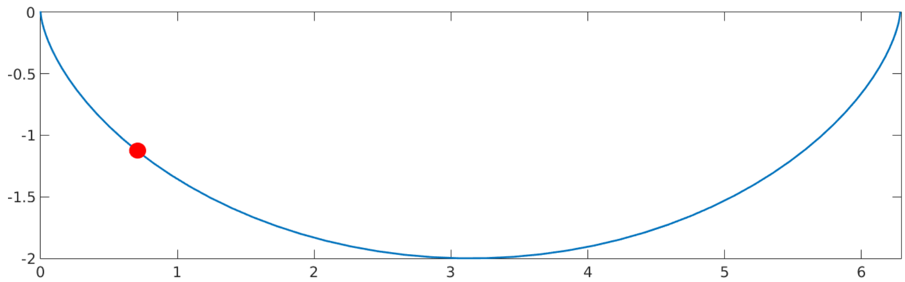


Figure 5.10: A cycloid.

is quite intricate, hence we will only discuss it in the next chapter. It can be proven that the equation of motion is really harmonic

$$\ddot{S} + \frac{g}{4R}S = 0 \tag{5.35}$$

where S stands for the arc length (measured from the apex of the cycloid).

5.1.5 Simple Pendulum Driven Harmonically at the Suspension Point

The analytic mechanics (see detail in the next chapter) gives opportunity to find the equations of motions for coupled oscillators. One of the most simple examples for coupling oscillators is a system that consists of a linear harmonic oscillator connected to the suspension point of a simple pendulum. Such a system is depicted in figure 5.11. Despite its simplicity, it generally does extremely complicated motion. The general behaviour of the motion is chaotic. *However, if the perturbation of the system in the equilibrium position is sufficiently small the response function of this system can be harmonic motion.* This approach is called a *small oscillation approximation*. In this case the equations of motions take the following form:

$$\ddot{x}_1 + l\ddot{\theta} + g\theta = 0 \tag{5.36}$$

$$\ddot{x}_1 + l\ddot{\theta} + \frac{k}{m}x_1 = 0 \tag{5.37}$$

After subtracting the two equations from each other we get:

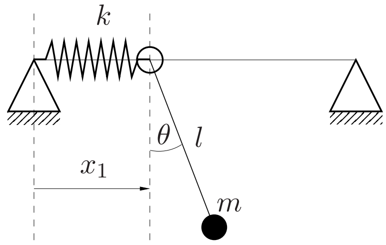


Figure 5.11: Harmonically driven pendulum.

$$x_1 = \frac{mg}{k}\theta$$

and if we substitute x_1 into equation (5.36) we get:

$$\ddot{\theta} + \frac{1}{\frac{m}{k} + \frac{l}{g}}\theta = 0$$

Hence, the angular frequency:

$$\omega = \frac{1}{\sqrt{\frac{m}{k} + \frac{l}{g}}}$$

We emphasized, that equations (5.36) and (5.37) do not generally characterise the motion. They are valid in the limit of small oscillation. The general equations will be given in the next chapter.

5.2 Damped Oscillations

In simple harmonic motion, we assume that no dissipative forces such as friction or viscous drag exist. Since the mechanical energy is constant, the oscillations continue for ever with constant amplitude. SHM is a simple model.The oscillations of a swinging pendulum or vibrating tuning fork gradually die out as energy is dissipated. The amplitude of each cycle is a little smaller than that of the previous cycle. This kind of motion is called damped oscillation.

Damping is not always a disadvantage. Thus the suspension system of a car includes shock absorbers that cause the vibration of the body to be quickly damped. The shock absorbers reduce the discomfort that passengers would otherwise experience due to the bouncing of an automobile as it travels along a bumpy road.

Damped oscillations are described by the following second order differential equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \tag{5.38}$$

When damping is not too great, the general solution of equations (5.38) has the form:

$$x(t) = A_0 e^{-\beta t}\cos(\omega t + \alpha)$$

The motion of the system can be considered as a harmonic oscillation of frequency ω with an amplitude varying according to the law $A(t) = A_0 e^{-\beta t}$.

5.3 Forced Oscillations and Resonance

When damping force is present, the only way to keep the amplitude of oscillations from diminishing is to replace the dissipated energy from some other source.

Forced oscillations occur when a periodic external driving force acts on a system that can oscillate. The frequency of the driving force does not have to match the natural frequency of the system. Ultimately, the system oscillates at the driving frequency, even if it is far from the natural frequency. However, the amplitude of the oscillations is generally quite small unless the driving frequency ω is close to the natural frequency ω_0.

When the driving force changes according to a harmonic law, the oscillations are described by the linear differential equation of the second order with constant coefficients

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0\cos(\omega t)$$

The solution of these equations are the following:

$$x(t) = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta\omega^2}}\cos\left(\omega t - \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

When a driving frequency is close to the natural frequency ω_0, the amplitude of the motion is maximum. This condition is called *resonance.*

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - 2\beta^2}$$

Large-amplitude vibration due to resonance can be dangerous in some situations. In the nineteenth century, bridges were sometimes set into resonant vibration when the cadence of marching soldiers matched a resonant frequency of the bridge. Soldiers were told to break step when crossing the bridge.

The resonance gives opportunity to amplify very small effects as the radiation pressure exerted upon any surface due to the exchange of momentum between the object and the electromagnetic field. Stochastic resonance (SR) is a phenomenon where a signal that is normally too weak to be detected by a sensor, can be boosted by adding white noise to the signal, which contains a wide spectrum of frequencies. The frequencies in the white noise corresponding to the original signal's frequencies will resonate with each other, amplifying the original signal while not amplifying the rest of the white noise

Chapter 6

Lagrangian Formalism

6.1 Some Comments on Analytical Mechanics

In most situations, the problems are not that simple to solve by means of dynamical and initial conditions, for example, a mass that is constrained to move on a spherical surface or a bead that slides on a wire. In these situations, not only the unknown form of the forces of constraints makes the problem difficult to solve, but using the standard coordinates may make it impossible to tackle the problem. Two different methods, *Lagrange's equations* and *Hamilton's equations* have been developed to handle such problems. These two techniques offer much ease in handling very difficult problems of a physical nature. First, these two techniques use *generalized coordinates*. That is, instead of being limited to the use of rectangular or polar coordinates and a like, any suitable quantity, such as velocity, linear momentum, angular momentum, is used in solving problems. Such generalized coordinates are usually denoted by q_k ($k = 1, 2, 3 \ldots n$). Furthermore, these techniques use energy approach, having the primary advantage of dealing with scalars, rather than vectors. In Lagrange's formalism the generalized coordinates used are position and velocity, resulting the second order differential equations. In Hamilton's formalism the generalized coordinates used are position and momentum, resulting in first-order linear differential equations.

If the system has n degree of freedom the Lagrange's equations are the following:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{6.1}$$

where $k = 1, 2, 3 \ldots n$, hence if the system has two degrees of freedom we get two Lagrange equations. For conservative fields

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q) \tag{6.2}$$

where T is the kinetic energy and V stands for the potential energy.

Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial V}{\partial q_k} \tag{6.3}$$

6.2 Lagrangians

6.2.1 Free particle

If a particle is moving in a zero potential ($V = 0$) the particle is called free. The Lagrangian of a free particle is the following:

$$L = T - V = \frac{m\dot{q}^2}{2} - 0$$

Hence,

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \quad \frac{\partial L}{\partial q} = -F \tag{6.4}$$

The Lagrange equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \dot{p} \tag{6.5}$$

$$\frac{\partial V}{\partial q} = 0 \tag{6.6}$$

Equating the right hand side of the equations we get:

$$\dot{p} = 0 \tag{6.7}$$

$$p = \text{COM} \tag{6.8}$$

COM means constant of motion.

6.2.2 Linear Oscillator

If a particle is moving in a parabolic potential ($V = kq^2/2$) it oscillates back and forth with simple harmonic motion. Such a system is called a linear oscillator and its Lagrangian takes the following form:

$$L = T - V = \frac{m\dot{q}^2}{2} - \frac{1}{2}kq^2 = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2}$$

$$\frac{\partial T}{\partial \dot{q}} = m\dot{q} \tag{6.9}$$

$$\frac{\partial V}{\partial q} = -m\omega^2 q = -kq \tag{6.10}$$

$\partial T/\partial \dot{q}$ is often called *generalised momentum* and $\partial V/\partial q$ is abbreviated as *generalised force*.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = m\ddot{q} \tag{6.11}$$

$$\frac{\partial V}{\partial q} = -m\omega^2 q \tag{6.12}$$

Equating the right hand side of the equations we get:

$$m\ddot{q} = -m\omega^2 q \tag{6.13}$$
$$\ddot{q} + \omega^2 q = 0 \tag{6.14}$$

6.2.3 Vertical Oscillation

Find the Lagrangian of a one dimensional oscillator if it moves in vertical direction in the Earth's gravitational field. At the initial time moment the spring is undeformed and the initial velocity of the particle is zero. At a given moment the particle is released. Let the vertical coordinate y be the generalised coordinate q. The Lagrangian of the system is:

$$L = \frac{m\dot{q}^2}{2} - \frac{1}{2}kq^2 - mg\left(q_{\max} - q\right) \tag{6.15}$$
$$L = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2} - \left(\frac{2m^2 g^2}{k} - mgq\right) \tag{6.16}$$

The maximum energy of the system

$$E_{\mathrm{TOT}} = \frac{2m^2 g^2}{k}$$

is constant. Any constant term of the Lagrangian can be omitted because its derivative will be identically zero, hence the Lagrangian takes the following form:

$$L = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2} + mgq$$

The equation of motion is

$$\ddot{q} + \omega^2 q - g = 0 \tag{6.17}$$
$$\ddot{q} + \frac{k}{m}\left(q - \frac{mg}{k}\right) = 0 \tag{6.18}$$

In a general constant force field (F) the lagrangian reads as:

$$L = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2} + Fq$$

6.2.4 Inclined Plane

One of the most interesting features of analytic formalism is that we do not have to define the constraint forces. The most trivial example is the inclined plane (see figure (6.1)). We introduce S as a generalized coordinate and we know that the gravitational field is conservative, hence the Lagrange function reads as:

$$L = T - V = \frac{1}{2}m\dot{S}^2 - (-mgy) \tag{6.19}$$
$$y = S\sin\theta \tag{6.20}$$

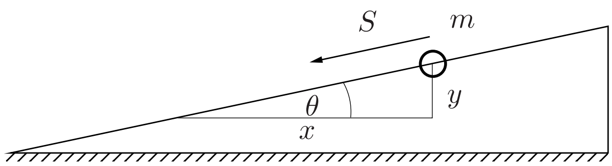


Figure 6.1: Inclined plane.

Hence

$$L = \frac{1}{2} m \dot{S}^2 + mg \sin(\theta) S \tag{6.21}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) = m\ddot{S} \tag{6.22}$$

$$\frac{\partial V}{\partial q} = mg \sin(\theta) \tag{6.23}$$

$$\cancel{m}\ddot{S} = \cancel{m}g \sin(\theta) \tag{6.24}$$

$$\ddot{S} = g \sin(\theta) \checkmark \tag{6.25}$$

As we can see only the gravitational force had to be taken into account. The constraint was manifested in the mathematical relationships between the S and y coordinates.

6.2.5 Double Inclined Plane

Let us consider a fixed inclined plane. In the surface of the fixed incline there is a similar, oppositely directed inclined plane with mass M in accordance with figure (6.2). A small object with mass m is placed on the vertical surface of the second incline. The friction force is negligibly small. Find the accelerations of the objects. In the framework of constarint forces this double inclined plane problem is a bit problematic. However, if we use the Lagrange function it becomes very simple. The small m accelerates

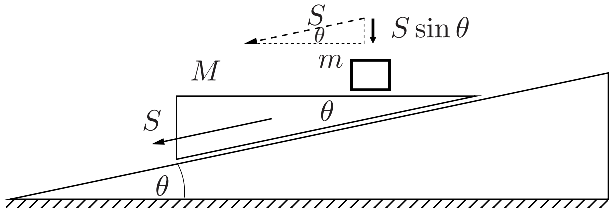


Figure 6.2: Smaller inclined plane slides down on the surface of a fixed inclined plane.

vertically downward because no force acts on the body in the horizontal direction. For this reason: $v_m = \dot{S} \sin\theta$.

$$L = \frac{1}{2} M \dot{S}^2 + \frac{1}{2} m \left(\dot{S} \sin\theta\right)^2 + S \sin\theta \,(m + M)\, g \tag{6.26}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) = M\ddot{S} + m\ddot{S}\sin^2\theta \tag{6.27}$$

$$\frac{\partial V}{\partial S} = (m+M)\, g\sin\theta \tag{6.28}$$

The accelerations of the objects are:

$$\ddot{S} = a_M = \frac{(m+M)\, g\sin\theta}{M + m\sin^2\theta} \tag{6.29}$$

$$\ddot{S}\sin\theta = a_m = \frac{(m+M)\, g}{\frac{M}{\sin^2\theta} + m} \tag{6.30}$$

It is advisable to check our results in the limit $M \longrightarrow \infty$ and $M \longrightarrow 0$:

$$\lim_{M\longrightarrow\infty} \ddot{S} = \frac{\left(\frac{m}{M}+1\right) g\sin\theta}{1+\frac{m}{M}\sin^2\theta} = g\sin\theta\checkmark$$

$$\lim_{M\longrightarrow 0}\left(\ddot{S}\sin\theta\right) = \frac{(m+0)\, g\sin^2\theta}{0+m\sin^2\theta} = g\checkmark$$

The results seem reasonable. In case of introducing a constraint force N between m and M the equations of motions are the following:

$$mg - N = ma_m$$
$$(Mg+N)\sin\theta = Ma_M$$
$$a_M\sin\theta = a_m$$

The solution of the system of equations yields:

$$a_m = \frac{(m+M)\, g}{m+\frac{M}{\sin^2\theta}}$$
$$N = m\left(g - \frac{(m+M)\, g\sin^2\theta}{m\sin^2\theta + M}\right)$$

It is useful to check N in the limit $\theta \longrightarrow 0$ and $\theta \longrightarrow \pi/2$:

$$\lim_{\theta\longrightarrow 0} N = m\left(g - \frac{(m+M)\, g\cdot 0}{m\cdot 0 + M}\right) = mg\checkmark$$

$$\lim_{\theta\longrightarrow \pi/2} N = m\left(g - \frac{(m+M)\, g\cdot 1}{m\cdot 1 + M}\right) = 0\checkmark$$

The N constraint force is generally less than mg.

6.3 Advanced Problems

6.3.1 Joined Cylinders

As it can be seen in figure (6.3) a cylinder is connected to a hanging cylinder with a rope that goes over a pulley. The cylinder on the left is rolling without slipping. Assume that the rope does not slip. What is the Lagrangian of the system? The

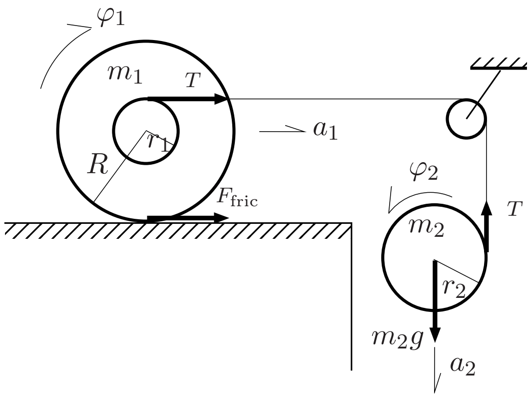


Figure 6.3: A mechanical system consists of rolling cylinders joined by an ideal rope.

system has two degrees of freedom (φ_1, φ_2), which can be seen in the Lagrangian.

$$L = \frac{3}{4} m_1 R^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \left[(R + r_1)\, \dot{\varphi}_1 + r_2 \dot{\varphi}_2 \right]^2 + \frac{1}{2}\frac{1}{2} m_2 r_2^2\, \dot{\varphi}_2^2$$
$$+ m_2 g \left[(R + r_1)\, \varphi_1 + r_2 \varphi_2 \right]$$

According to the Lagrange equations we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) = \frac{3}{2} m_1 R^2\, \ddot{\varphi}_1 + \frac{1}{2} m_2 (R + r_1)^2\; 2\, \ddot{\varphi}_1 + \frac{1}{2} m_2 2 r_1 (R + r_1)\, \ddot{\varphi}_2 \tag{1}$$

$$\frac{\partial L}{\partial \varphi_1} = m_2 g (R + r_1) \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) = \frac{1}{2} m_2 2 r_2 (R + r_1)\, \ddot{\varphi}_1 + \frac{1}{2} m_2 2 r_2^2\, \ddot{\varphi}_2 + \frac{1}{2}\frac{1}{2} m_2 r_2^2 2\, \ddot{\varphi}_2 \tag{3}$$

$$\frac{\partial L}{\partial \varphi_2} = m_2 g r_2 \tag{4}$$

The equations of motion are the following:

$$\frac{3}{2} m_1 R^2\, \ddot{\varphi}_1 + m_2 (R + r_1)^2\; \ddot{\varphi}_1 + m_2 r_1 (R + r_1)\, \ddot{\varphi}_2 = m_2 g (R + r_1) \tag{I}$$

$$m_2 r_2 (R + r_1)\, \ddot{\varphi}_1 + m_2 r_2^2\, \ddot{\varphi}_2 + \frac{1}{2} m_2 r_2^2\, \ddot{\varphi}_2 = m_2 g r_2 \tag{II}$$

After solving the equations (I) and (II) we get:

$$\ddot{\varphi}_1 = \frac{2(R+r_1)m_2 g}{m_2(R+r_1)^2 + 9m_1R^2} \tag{6.31}$$

$$\ddot{\varphi}_2 = \frac{6R^2 m_1 g}{r_2\left(2m_2(R+r_1)^2 + 9m_1R^2\right)} \tag{6.32}$$

The accelerations of the CM of the cylinders are:

$$\begin{aligned} a_1 &= R\alpha_1 & /\alpha_1 = \ddot{\varphi}_1/ \\ a_2 &= (R+r_1)\alpha_1 + r_2\alpha_2 & /\alpha_2 = \ddot{\varphi}_2/ \end{aligned}$$

The Newton's equations of the system are the following:

$$T + F_{\text{fric}} = m_1 R\alpha_1 \tag{1}$$

$$Tr_1 - F_{\text{fric}}R = \frac{1}{2}m_1R^2\alpha_1 \tag{2}$$

$$m_2 g - T = m_2\left[(R+r_1)\alpha_1 + r_2\alpha_2\right] \tag{3}$$

$$Tr_2 = \frac{1}{2}m_2 r_2^2\alpha_2 \tag{4}$$

The constraint forces (T and F_{fric}) are completely missing from the Lagrange function, however, in the Newton's equations they play a fundamental role.

6.3.2 Lagrangian of a Disk inside a Semicircular Well

If we recall our former problem (disk inside a semicircular well) in the framework of analytical mechanics the equation of motion can be easily deduced. The problem was worded as follows: a cylinder of mass m and radius r rolls without slipping inside a semicircular track of radius R that occupies a vertical plane as it can be seen in figure (6.4). Let φ represent the angle of rotation of the cylinder, and let ϕ measure the angle between the centre of the semicircle and the centre of the cylinder, with respect to the vertical. Now we seek the Lagrangian and the equation of motion of the system.

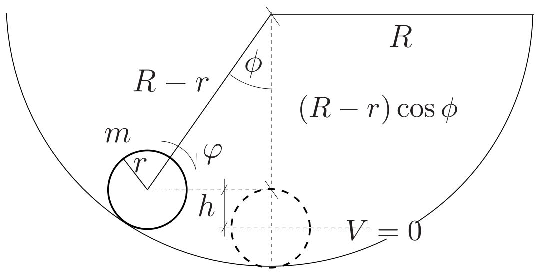


Figure 6.4: A disk rolls back and forth in a semicircular well.

As we have seen before the total energy of the system can be expressed in the following form:

$$E_{\text{tot}} = mg\left(R-r\right)\left(1-\cos\left(\phi\right)\right) + \frac{1}{2}\frac{3}{2}m\left(R-r\right)^2\dot{\phi}^2$$

Since we can omit the constant part of the potential energy $(mg\left(R-r\right))$ the Lagrangian of the system $(L = T - V)$ is as follows:

$$L = \frac{1}{2}\frac{3}{2}m\left(R-r\right)^2\dot{\phi}^2 + mg\left(R-r\right)\cos\left(\phi\right)$$

Taking into account the left hand side and right hand side of the Euler-Lagrange equation we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{3}{2}m\left(R-r\right)^2\ddot{\phi} \tag{6.33}$$

$$\frac{\partial L}{\partial\phi} = -mg\left(R-r\right)\sin\left(\phi\right) \tag{6.34}$$

Finally, the equation of motion is:

$$\frac{3}{2}\cancel{m}\left(R-r\right)^{\cancel{2}}\ddot{\phi} = -\cancel{m}g\cancel{\left(R-r\right)}\sin\left(\phi\right) \tag{6.35}$$

$$\ddot{\phi} + \frac{2g}{3\left(R-r\right)}\sin\left(\phi\right) = 0 \tag{6.36}$$

In small angle approximation $\sin\left(\phi\right)$ can be approximated by ϕ. Under such circumstances the system performs (approximately) simple harmonic motion.

$$\ddot{\phi} + \frac{2g}{3\left(R-r\right)}\phi = 0 \tag{6.37}$$

6.3.3 Isochronism of the Cycloidal Motion

A heavy particle constrained to frictionless oscillation under gravity along the arc of a cycloid. Isochronism means that the time period is independent of amplitude. A

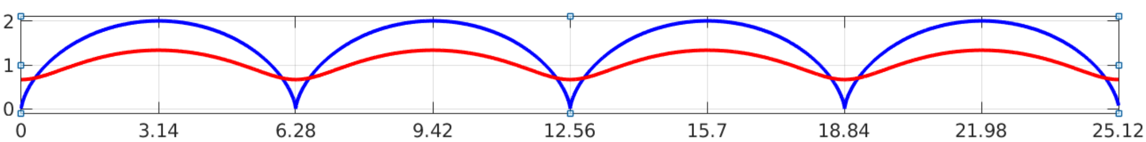


Figure 6.5: The curve of a cycloid and the curve of an inner point of a wheel.

cycloid is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slipping (see figure (6.5)). The cycloid through the origin, with a horizontal base given by the line $y = 0$, this line also known as the x-axis, generated by a circle of radius r rolling over the "positive" side of the base ($y \geq 0$), consists of the points (x, y), with

$$x = R\varphi - R\sin\varphi \tag{1}$$
$$y = R - R\cos\varphi \tag{2}$$

The entire arc length of the cycloid can be calculated by:

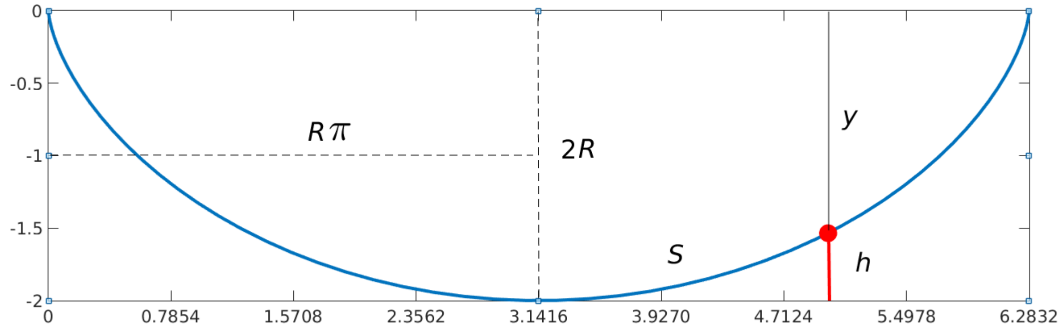


Figure 6.6: We have depicted a mirror image ($y' = -y$, $x' = x$) of the cycloid over a horizontal line. The value of R is chosen to be one unit.

$$S_{2\pi} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2} d\varphi = 4R\left[1 - \sqrt{\frac{1+\cos\varphi}{2}}\right]_0^{2\pi} = 8R$$

The arc length as a function of the angular coordinate (φ)

$$S(\varphi) = 4R\left[1 - \sqrt{\frac{1+\cos\varphi}{2}}\right]$$

The arc length S of the cycloid starting from the deepest point (see figure (6.6)) takes the following form:

$$S = 4R - 4R\left\{1 - \sqrt{\frac{1+\cos\varphi}{2}}\right\}$$
$$S = 4R\sqrt{\frac{1+\cos\varphi}{2}}$$
$$S^2 = 16R^2\frac{1+\cos\varphi}{2}$$
$$\frac{S^2}{8R^2} = 1 + \cos\varphi \tag{6.38}$$

From the parametric equation of the cycloid we can determine $\cos\varphi$ as a function of the vertical position (y)

$$\frac{y}{R} = 1 - \cos\varphi$$
$$\cos\varphi = 1 - \frac{y}{R}$$

Substituting $\cos\varphi = 1 - y/R$ into equation (6.38) we get h as a function of the arc-length:

$$\frac{S^2}{8R^2} = 1 + 1 - \frac{y}{R}$$
$$\frac{S^2}{8R^2} = 2 - \frac{y}{R}$$
$$\frac{S^2}{8R} = 2R - y$$
$$\frac{S^2}{8R} = h$$

The Lagrangian of the system takes the following form:

$$L = \frac{1}{2}m\dot{S}^2 - mgh$$
$$L = \frac{1}{2}m\dot{S}^2 - mg\frac{S^2}{8R}$$

According to the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{S}}\right) = m\ddot{S} \tag{6.39}$$
$$\frac{\partial V}{\partial S} = -\frac{mg}{4R}S \tag{6.40}$$

Equating the right hand sides of the equations and the equation is set to zero

$$\ddot{S} + \frac{g}{4R}S = 0 \tag{6.41}$$

According to equation (6.42) the particle oscillates harmonically with time period

$$T = 2\pi\sqrt{\frac{4R}{g}}$$

The slope of the tangent line in the position $x = 0$ or $x = 2\pi$ infinity. Mathematically it means that the tangent lines of the cycloid in these positions are vertical lines. Physically it means that in these positions the accelerations must be g.

$$a = A\omega^2 = \cancel{4R}\frac{g}{\cancel{4R}} = g\checkmark$$

6.3.4 Inverse Oscillator

If the particle slides down a convex cycloid, the equation takes the following form:

$$\ddot{S} - \frac{g}{4R}S = 0 \tag{6.42}$$

It is an ordinary differential equation which can be solved in many ways, its solution is:

$$S(t) = A\exp\left(\frac{g}{4R}t\right) + B\exp\left(-\frac{g}{4R}t\right)$$

Where A and B are constants to be determined by the initial boundary conditions in the problem. The initial condition is $S(0) = S_0$, where S_0 the arc length is measured from the apex (highest point) of the cycloid, and $\dot{S}_0 = 0$.

$$S(t) = \frac{S_0}{2}\left(\exp\left(\frac{g}{4R}t\right) + \exp\left(-\frac{g}{4R}t\right)\right)$$

$$t = \sqrt{\frac{4R}{g}}\cosh^{-1}\left(\frac{4R - S_0}{S_0}\right)$$

6.3.5 A system consisting of five bodies

Find the Lagrangian for the following system (see figure (6.7)). Assume there is no slipping.

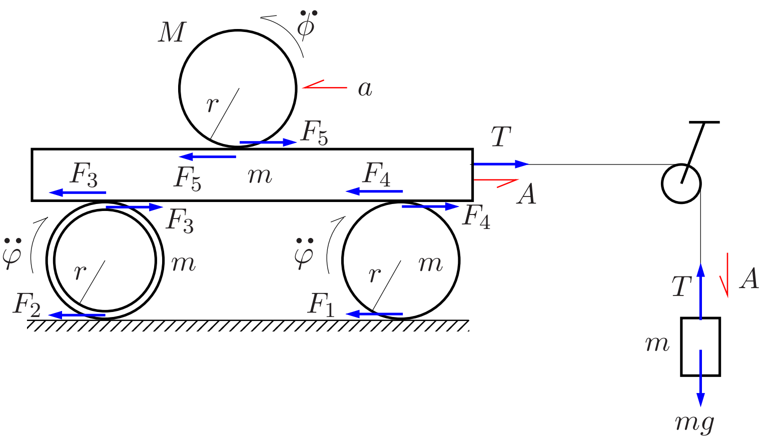


Figure 6.7: A mechanical system consists of two blocks and three rolling objects.

The velocity of the central mass of the upper cylinder compared to the initial observer (fixed to the ground) is $v_M = \left(2r\dot{\varphi} - r\dot{\phi}\right)$. Hence the Lagrangian of the system takes the following form:

$$L = \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{3}{4}mr^2\dot{\varphi}^2 + \frac{1}{2}m\left(2r\dot{\varphi}\right)^2 + \frac{1}{2}M\left(2r\dot{\varphi} - r\dot{\phi}\right)^2 + \frac{1}{2}\frac{1}{2}Mr^2\dot{\phi}^2 + \frac{1}{2}m\left(2r\dot{\varphi}\right)^2 + mg2r\varphi$$

The equation can be simplified to the following form:

$$L = \frac{23}{4}mr^2\dot{\varphi}^2 + 2Mr^2\dot{\varphi}^2 - 2Mr^2\dot{\varphi}\dot{\phi} + \frac{3}{4}Mr^2\dot{\phi}^2 + 2mgr\varphi$$

According to the Lagrange equations we get:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{23}{2}mr^2\ddot{\varphi} + 4Mr^2\ddot{\varphi} - 2Mr^2\ddot{\phi} \tag{1}$$
$$\frac{\partial L}{\partial \varphi} = 2mgr \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = -2Mr^2\ddot{\varphi} + \frac{3}{2}Mr^2\ddot{\phi} \tag{3}$$
$$\frac{\partial L}{\partial \phi} = 0 \tag{4}$$

Equating the right-hand sides of the pairs of equations we get:

$$\frac{23}{2}mr^2\ddot{\varphi} + 4Mr^2\ddot{\varphi} - 2Mr^2\ddot{\phi} = 2mgr \tag{I}$$
$$-2Mr^2\ddot{\varphi} + \frac{3}{2}Mr^2\ddot{\phi} = 0 \tag{II}$$

The solution of the equations for the accelerations is:

$$2r\ddot{\varphi} = \frac{24mg}{69m+8M} \tag{6.43}$$
$$r\ddot{\phi} = \frac{16mg}{69m+8M} \tag{6.44}$$
$$\left(2r\ddot{\varphi} - r\ddot{\phi}\right) = \frac{8mg}{69m+8M} \tag{6.45}$$

The acceleration of the central mass of the upper cylinder compared to the ground is $a_M = (A-a)$, hence the Newton's equations of the systems are the following:

$$mg - T = mA \tag{1}$$
$$T - (F_3 + F_4 + F_5) = mA \tag{2}$$
$$F_4 - F_1 = \frac{mA}{2} \tag{3}$$
$$F_4\cancel{r} + F_1\cancel{r} = \frac{1}{2}m\cancel{r}^{\cancel{2}}\frac{A}{2\cancel{r}} \tag{4}$$
$$F_3 - F_2 = \frac{mA}{2} \tag{5}$$
$$F_3\cancel{r} + F_2\cancel{r} = m\cancel{r}^{\cancel{2}}\frac{A}{2\cancel{r}} \tag{6}$$
$$F_5 = M(A-a) \tag{7}$$
$$F_5 r = \frac{1}{2}Mr^2\frac{a}{r} \tag{8}$$

After simplifications the equation takes the following form:

$$mg - T = mA \tag{1}$$

$$T - (F_3 + F_4 + F_5) = mA \tag{2}$$

$$F_4 - F_1 = \frac{mA}{2} \tag{3}$$

$$F_4 + F_1 = \frac{mA}{4} \tag{4'}$$

$$F_3 - F_2 = \frac{mA}{2} \tag{5}$$

$$F_3 + F_2 = \frac{mA}{2} \tag{6'}$$

$$F_5 = M\,(A - a) \tag{7}$$

$$F_5 r = \frac{1}{2} M r^2 \frac{a}{r} \tag{8}$$

Interestingly, the equations show that the static friction force F_2 vanishes. The solution of the equations for the accelerations is:

$$A = \frac{24mg}{69m + 8M} \tag{6.46}$$

$$A - a = \frac{8mg}{69m + 8M} \tag{6.47}$$

$$\tag{6.48}$$

6.3.6 Atwood machine with spring

The ideal Atwood Machine consists of two objects of mass m_1 and m_2, connected by an inextensible massless string over an ideal massless pulley. We replaced one of the massive objects with an ideal spring (k), and we assume that the pulley has some mass M. The system can be seen in figure (6.8).

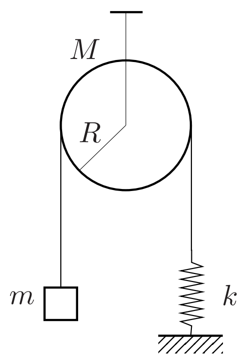


Figure 6.8: A mechanical system consists of a spring, a fixed pulley and a suspended body.

The Lagrangian of the system:

$$L = \frac{1}{2} m \left(R\dot{\varphi}\right)^2 + \frac{1}{2}\frac{1}{2} M \left(R\dot{\varphi}\right)^2 + mgR\varphi - \frac{1}{2} k R^2 \varphi^2$$

According to the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{1}{2}mR^2\,2\,\ddot{\varphi} + \frac{1}{4}MR^2\,2\,\ddot{\varphi} \tag{1}$$

$$\frac{\partial L}{\partial \varphi} = mgR - \frac{1}{2}kR^2\,2\varphi \tag{2}$$

we get:

$$\ddot{\varphi} + \frac{k}{m+\frac{M}{2}}\varphi - \frac{mg}{\left(m+\frac{M}{2}\right)R} = 0 \tag{3}$$

$$\ddot{\varphi} + \frac{2k}{2m+M}\left(\varphi - \frac{mg}{kR}\right) = 0 \tag{3'}$$

$$\ddot{\phi} + \frac{2k}{2m+M}\phi = 0 \tag{3''}$$

The time period of the system is:

$$T = 2\pi\sqrt{\frac{2m+M}{2k}}$$

6.3.7 Cylinder with three different springs

A mechanical system consists of three different springs, characterised by different spring constants (k, $2k$, and $3k$) and a rolling cylinder as it can be seen in figure (6.9). Find the time period of the following system in the small oscillation approximations. Assume that the cylinder rolls without slipping.

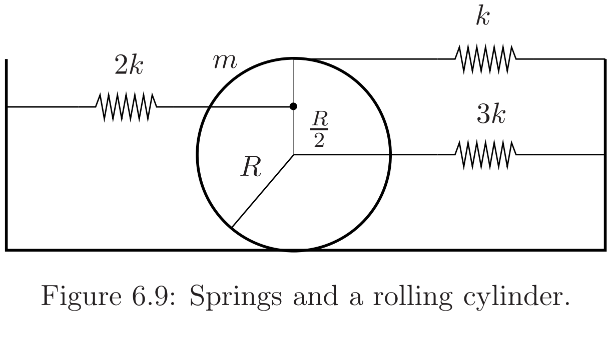


Figure 6.9: Springs and a rolling cylinder.

The Lagrangian of the system is:

$$L = \frac{3}{4}mR^2\dot{\varphi}^2 - \frac{1}{2}2k\left(R + R/2\right)^2\varphi^2 - \frac{1}{2}\,3k\,R^2\varphi^2 - \frac{1}{2}k\left(2R\varphi\right)^2 \tag{1}$$

$$L = \frac{3}{4}mR^2\dot{\varphi}^2 - \frac{23}{4}kR^2\varphi^2 \tag{1'}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{3}{2}mR^2\ddot{\varphi}$$
$$\frac{\partial L}{\partial \varphi} = -\frac{23}{2}kR^2\varphi$$
$$\frac{3}{2}mR^2\ddot{\varphi} = -\frac{23}{2}kR^2\varphi$$

$$\ddot{\varphi} + \frac{23k}{3m}\varphi = 0 \tag{6.49}$$

Hence, the time period takes the following form:

$$T = 2\pi\sqrt{\frac{3m}{23k}}$$

6.3.8 A simple pendulum driven harmonically

The analytic mechanics gives opportunity to find the equations of motions for a system that consists of a linear harmonic oscillator connected to the suspension point of a simple pendulum (see figure (6.10)). A simple pendulum driven harmonically at the suspension point. Despite its simplicity, it generally does extremely complicated motion. The general behaviour of the motion is chaotic. The Lagrangian of the system reads as:

$$L = \frac{1}{2}m\dot{x}_1 + \frac{1}{2}m\left(2\dot{x}_1\dot{\theta}l\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta - \frac{kx_1^2}{2}$$

The equations of motion are the following:

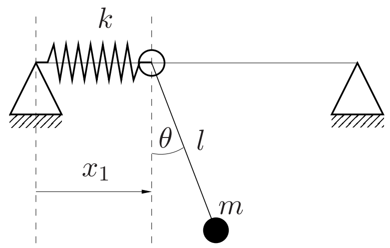


Figure 6.10: A simple pendulum driven at the suspension point.

$$\ddot{x}_1\cos\theta - \dot{x}_1(\sin\theta)\,\dot{\theta} + l\ddot{\theta} + g\sin\theta = 0 \tag{6.50}$$
$$\ddot{x}_1 + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta + \frac{k}{m}x_1 = 0 \tag{6.51}$$

Since exact solutions can not be determined via algebraic methods numerical methods are used to solve these nonlinear differential equations. However, as the perturbation of the system in the equilibrium position is sufficiently small, the response function of this system can be harmonic motion. We use the method of linearization. In mathematics, linearization is finding the linear approximation to a function at a given point. In the study of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point of a system of nonlinear differential equations or discrete dynamical systems. Due to linearization $g\sin\theta \approx g\theta$ and because of the small oscillation approximation $\cos\theta \approx 1$ and those terms which do not include g but include sin functions are vanishing ($\sin\theta \approx 0$). After linearization the former equations take the following form:

$$\ddot{x}_1 + l\ddot{\theta} + g\,\theta = 0 \tag{6.52}$$

$$\ddot{x}_1 + l\ddot{\theta} + \frac{k}{m}x_1 = 0 \tag{6.53}$$

6.3.9 Double Pendulum

In physics a double pendulum is a pendulum with another pendulum attached to its end (see figure (6.11)), and is a simple physical system that exhibits rich dynamic behaviour with a strong sensitivity to initial conditions. The motion of a double pendulum is governed by a set of coupled ordinary differential equations and is chaotic. The Lagrangian of the system:

$$L = \frac{1}{2}m_1 l_1^2\,\dot{\theta}_1^2 + \frac{1}{2}m_2 l_1^2\,\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2\,\dot{\theta}_2^2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\,\dot{\theta}_1\dot{\theta}_2 + \tag{6.54}$$

$$(m_1 + m_2)\,g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 \tag{6.55}$$

The equations of motions are the following:

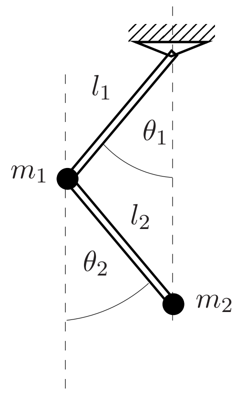


Figure 6.11: A double pendulum.

$$m_2 l_2^2\,\ddot{\theta}_2 + m_2 l_1 l_2\,\ddot{\theta}_1\,\cos(\theta_1 - \theta_2) - m_2 l_1 l_2\,\dot{\theta}_1^2\,\sin(\theta_1 - \theta_2) + l_2 m_2 g \sin\theta_2 = 0 \tag{6.56}$$

$$m_2 l_2\,\ddot{\theta}_2 + m_2 l_1\,\ddot{\theta}_1\,\cos(\theta_1 - \theta_2) - m_2 l_1\,\dot{\theta}_1^2\,\sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0 \tag{6.57}$$

6.4 The Phase Plane

Let us take again the harmonic oscillator, $a + \omega^2 x = 0$. As we discussed earlier the total energy of the system is:

$$E = \frac{1}{2}m\dot{x}^{\,2} + \frac{1}{2}kx^2 \tag{6.58}$$

Energy is a function of two variables, x and $\dot{x}$. The plane of these coordinates is called the *phase plane*. It is traditional to draw x horizontally and $\dot{x}$ vertically.

Let us derive the contours of constant energy. Since the position-time and velocity-time characteristics of the motion are:

$$x(t) = A\cos(\omega t + \gamma) \tag{6.59}$$

$$\dot{x}(t) = -A\omega\sin(\omega t + \gamma) \tag{6.60}$$

In order to eliminate the explicit time dependence of the solutions we isolate the trigonometric functions as follows:

$$\frac{x}{A} = \cos(\omega t + \gamma) \tag{6.61}$$

$$\frac{\dot{x}}{A\omega} = -\sin(\omega t + \gamma) \tag{6.62}$$

The equations are raised to the power of two and after we have added them the time dependence vanishes. The elimination of time dependence yields mathematically an ellipse.

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega^2} = 1 \tag{6.63}$$

The total energy of the system can be expressed with the amplitude $E = kA^2/2$ and with the maximum velocity $E = mA^2\omega^2/2$ as well. For this reason A^2 and $A^2\omega^2$ can be expressed with the total energy E.

$$A^2 = 2E/k \tag{6.64}$$

$$A^2\omega^2 = 2E/m \tag{6.65}$$

Contours of constant energy are *ellipses* on the phase plane. Let the generalised coordinate be $q = x$ and after substituting $2E/k$ and $2E/m$ into equation (6.63) the ellipse takes the following form:

$$\frac{q^2}{2E/k} + \frac{\dot{q}^2}{2E/m} = 1 \tag{6.66}$$

So the amplitude of the particle $A = \sqrt{2E/k}$ is the semi-major and $A\omega = \sqrt{2E/m}$ is the semi-minor axis of the ellipse. According to these formulas for a given energy, small k means big swing and small m means large velocity. As time increases, the point $(q(t), v(t))$ traces out this ellipse. Since the ellipse depends on the net energy, we get a family of nested non-intersecting curves. This is called *phase diagram* of this system. In a phase diagram, trajectories move to the right above the horizontal axis, and to the left below it.

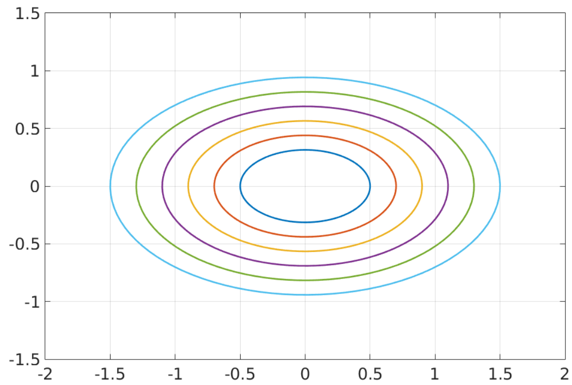


Figure 6.12: Contours of constant energy on the phase plane.

6.5 The Hamilton's equations

In Hamiltonian mechanics, a classical physical system is described by a set of canonical coordinates (p, q). The following equations are called *Hamilton's canonical equations.*

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \quad k = 1, 2, 3 \ldots n \tag{6.67}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad k = 1, 2, 3 \ldots n \tag{6.68}$$

Such formalism for a system of n degree of freedom leads to $2n$ first-order differential equations. These 2n first-order equations are much easier to solve than n second-order differential equations.

Another useful property of the Hamilton's formalism is that the Hamilton-function, unlike the Lagrangian, can be treated as the total energy of the system.

$$H(q, p) = T + V$$

The Lagrangian and Hamiltonian approaches provide the groundwork for deeper results in the theory of classical mechanics, and for formulations of quantum mechanics. In quantum mechanics, a Hamiltonian is an operator corresponding to the total energy of the system in most of the cases. It is usually denoted by H, also $\hat{H}$. Its spectrum is the set of possible outcomes when one measures the total energy of a system. Because of its close relation to the time-evolution of a system, it is of fundamental importance in most formulations of quantum theory.

6.5.1 Hamiltonian of a one-dimensional harmonic oscillator

In case of simple harmonic motion as time goes by, the phase point $(q(t), p(t))$ traces out an ellipse on the phase plane. We note if we use canonical coordinates that the mathematical form of this ellipse takes a slightly different form compared to equation (6.66). In order to make the linear momentum p appear in contour of energy we should multiply the second term in equation (6.66) by m^2/m^2 as follows:

$$\frac{q^2}{2E/k} + \frac{\dot{q}^2}{\frac{2}{m}E}\left(\frac{m^2}{m^2}\right) = 1 \tag{6.69}$$

$$\frac{q^2}{2E/k} + \frac{p^2}{2mE} = 1 \quad /p^2 = m^2\dot{q}^2 \tag{6.70}$$

$$\frac{q^2}{\frac{2}{m\omega^2}E} + \frac{p^2}{2mE} = 1 \quad /k = m\omega^2/ \tag{6.71}$$

The Hamiltonian of the system is equal to the net energy hence it is equivalent to E appearing in equation (6.71).

$$H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

The Hamilton's equations are as follows:

$$\dot{p} = -m\omega^2 q \tag{6.72}$$

$$\dot{q} = \frac{p}{m} \tag{6.73}$$

6.5.2 Hamiltonian of a Simple Pendulum

The simple pendulum is the prototype of one-dimensional nonlinear systems. The actual solutions are complicated involving the elliptic integrals and Jacobian elliptic functions.

Since the length of the pendulum l = constant, the system has only one degree of freedom, namely the angle θ.

$$p_\theta = ml^2\dot{\theta} \tag{6.74}$$

$$H(p_\theta, \theta) = \frac{p_\theta^2}{2ml^2} + mgl\,(1 - \cos\theta) \tag{6.75}$$

The phase space is defined by variables (p_θ, θ). Note that $I = ml^2$ is the moment of inertia of the system and further the angular momentum $L = I\omega$ both defined with respect to an axis of rotation. In this case it is the line passing through the support, perpendicular to the plane of vibration (rotation) or the pendulum.

While the Newtonian equation of motion is of second order,

$$l\ddot{\theta} + g\sin\theta = 0 \tag{6.76}$$

and the first order evolution (Hamilton's) equations are given by:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \tag{6.77}$$

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \tag{6.78}$$

If we neglect any dissipative forces the energy of the system is conserved. Hence the total energy of the system is time independent. Such systems are also called autonomous systems.

$$E = \frac{m}{2}l^2\dot{\theta}^2 + mgl\,(1-\cos\theta) \tag{6.79}$$

What do we expect for the motion? Well, if the energy E is less than a certain critical value ($2mgl$), then the pendulum will just swing back and forth. This kind of periodic motion is called libration. In contrast, if E is greater than the critical value, the pendulum will swing around and around. This kind of periodic motion is called rotation. If the energy is just equal to the critical value, there will be two

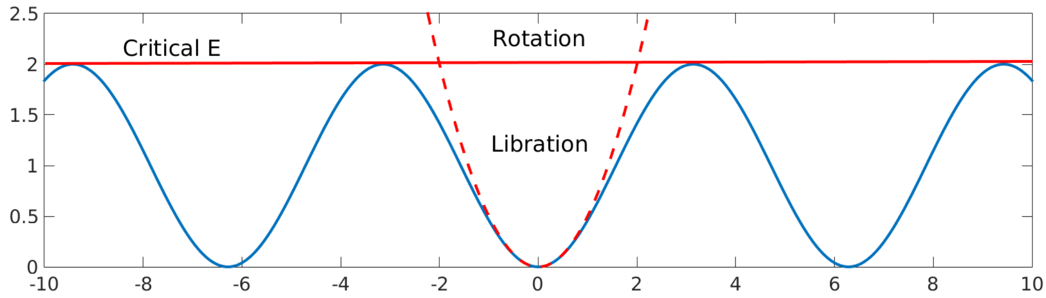


Figure 6.13: Critical energy, libration and rotation.

possibilities. If the pendulum starts out in motion, it will approach its vertical position ever more closely, without reaching it in any finite time. Or, the pendulum could start out perched exactly in the vertical position. It will remain there indefinitely. If the energy is zero, the pendulum just hangs straight down.

6.5.3 Phase Portrait

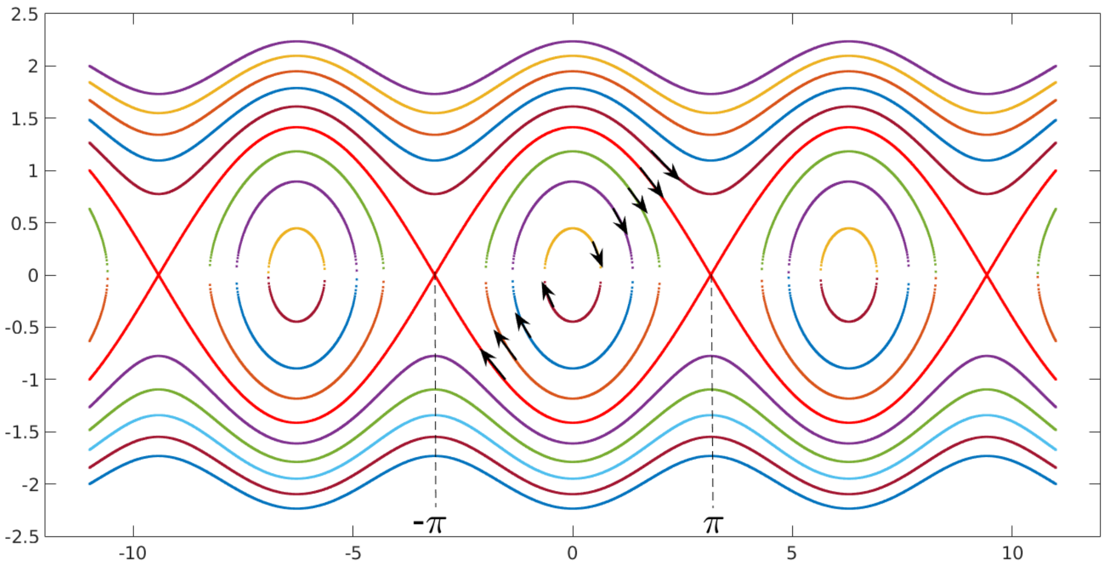


Figure 6.14: Phase Portrait of the simple pendulum.

An interesting way to view the motion of the pendulum is to plot the angular velocity versus the angle, as time goes on. You end up with several possibilities, depending on the energy. Some characteristic ones are shown in the Fig.(6.14).

The oval-shaped trajectories in the middle correspond to the librations, while the red one with intersection points corresponds to motions with energy equal to the critical energy. Such a trajectory is called a separatrix, because it separates regions with trajectories having different character. The trajectories outside this correspond to rotations.

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